

MISCELLANEOUS TRACTS

ON

Some curious, and very interesting SUBJECTS

IN

MECHANICS, PHYSICAL-ASTRONOMY, and SPECULATIVE MATHEMATICS;

WHEREIN,

The Precession of the EQUINOX, the Nutation of the EARTH'S AXIS, and the Motion of the MOON in her ORBIT, are determined.

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And

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L O N D O N,

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MDCCLVII.

MISCELLANEOUS

TRAGEDIES

1. The Tragedy of Hamlet, Prince of Denmark

2. The Tragedy of Othello, the Moor of Venice

3. The Tragedy of King Lear

4. The Tragedy of Macbeth

5. The Tragedy of Antony and Cleopatra

6. The Tragedy of Coriolanus

7. The Tragedy of Timon of Athens

8. The Tragedy of Cymbeline

9. The Tragedy of Troilus and Cressida

10. The Tragedy of Julius Caesar

11. The Tragedy of Caesar and Pompey

12. The Tragedy of the Two Gentlemen of Verona

13. The Tragedy of the Merchant of Venice

14. The Tragedy of The Winter's Tale

TO THE
RIGHT HONOURABLE
THE EARL OF
MACCLESFIELD, &c.

PRESIDENT of the ROYAL SOCIETY.

MY LORD,

WHATEVER Lustre, to the public Eye, Works of Learning may derive from the Patronage of the Great, it is to your Lordship's personal Acquirements, and extensive Knowledge in the Mathematical Sciences, that my Ambition of desiring leave to prefix your Name to this Performance, is to be imputed: And indeed, My Lord, an Author's natural Partiality permits me not to hope, or wish, that any thing these sheets contain, will meet with a more general Approbation, than what is due to the Propriety of their being inscribed to THE EARL OF MACCLESFIELD.

DEDICATION.

Were your Character, My Lord, less conspicuous and distinguish'd, the Obligations I have to your Lordship's Goodness, would, alone, be Motives sufficient to make me gladly embrace this Opportunity of publicly expressing my warmest Gratitude, and of testifying the perfect Esteem, and profoundest Deference, with which I am,

MY LORD,

Your Lordship's

most Obliged,

and ever Obedient

Humble Servant

Thomas Simpson.

P R E F A C E.

THE TRACTS, or PAPERS composing the WORK here offered to the Publick, were drawn up at several, distant times, and upon different occasions; either, with a view to clear up, or settle some difficult or controverted point in Astronomy, to shew the conformity of Theory with Observations; or to extend and facilitate the analytic-method of computation, by some improvements and applications, that have not at all, or but slightly, been touched upon, at least by any English Author.

The first of these PAPERS, which is one of the most considerable in the whole work, is concerned in determining the Precession of the Equinox, and the various inequalities thereof, with the different motions of nutation of the Earth's Axis, arising from the attraction of the sun and moon; wherein the late important discovery of Dr. Bradley, relating to an apparent motion of the Fix'd Stars, unknown to former Astronomers, is shewn to be intirely consistent with the Theory of Gravitation.

—This piece was drawn up about five years ago, in consequence of another on the same subject, by M. Silvabelle (a French Gentleman) then delivered to me, for my opinion, since printed in the Philosophical Transactions.—Tho' I have particular reasons for mentioning this circumstance, I would not be thought to insinuate here, that my opinion had any weight with Those to whom the publication of that paper was owing: I have, indeed, no reason to believe it.—Tho' the author thereof had gone through one part of the subject with success and perspicuity, and though his conclusions were found perfectly conformable to Dr. Bradley's observations, He nevertheless appeared (and still appears) to me: to have greatly failed in a very material, and indeed the only very difficult part, that is, in the determination of the momentary alteration of the position of the earth's axis, caused by the forces of the Sun and Moon; of which forces, the quantities, but not the effects, are truly investigated.

The Second Paper, contains the investigation of an easy, and very exact method, or rule, for finding the place of a Planet in its Orbit, from a correction of Dr. Ward's circular hypothesis; by means of certain Equations applied to the motion about the upper focus of the ellipsis. From whence that table of Dr. Halley's, entitled, *Tabula pro expediendo calculo AEquationis centri Lunæ*, may be very readily constructed.

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structed.—By this method, the result, even in the orbit of Mercury, may be found within a second of the truth, without repeating the operation.

The Third, shews the manner of transferring the motion of a Comet from a parabolic, to an elliptic Orbit; being of great use, when the observed Places of a (new) Comet, are found to differ sensibly from those computed on the hypothesis of a parabolic orbit.

The Fourth, is an attempt to shew, from mathematical principles, the advantage arising by taking the mean of a number of observations, in practical Astronomy; wherein the odds that the result, this way, is more exact, than from one single observation, is evinced, and the utility of the method in practice, clearly made to appear.—A part of this, and of the 7th paper, is inserted in the XLIX volume of the Philosophical Transactions; but the farther improvements here added, will (I hope) be a sufficient apology for my printing the whole again, in this work.

The Fifth, contains the determination of certain Fluents, and the resolution of some very useful Equations, in the higher orders of fluxions, by means of the measures of angles and ratios, and the right sines, and versed sines of circular arcs.

The Sixth, treats of the resolution of algebraical equations, by the method of surd-divisors; wherein the grounds of that method, as laid down by Sir Isaac Newton, are investigated and explained.

The Seventh, exhibits the investigation of a general rule for the resolution of Isoperimetrical Problems of all orders, together with some examples of the use and application of the said rule.

The Eighth (and last) Part, comprehends the resolution of some general, and very interesting problems, in mechanics and physical Astronomy; wherein, among other particulars, the principal parts of the third, and ninth sections of the first Book of Sir Isaac Newton's Principia, are demonstrated, in a new, and very concise manner.—But what, I apprehend, may best recommend this part of the work, is the application of the general equations therein derived, to the determination of the lunar Orbit: In which I have exerted my utmost endeavours to render the whole intelligible even to Those who have arrived but to a tolerable proficiency in the higher geometry.

The greater part of what is here delivered on this subject, was drawn up in the year 1750, agreeably to what is intimated at the conclusion of my Doctrine of Fluxions, where the general equations are also given. The famous objection, about that time made to Sir Isaac Newton's general Law of Gravitation, by that eminent mathematician M. Clairaut,

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of the Royal Academy of Sciences at PARIS, was a motive sufficient to induce me (among many Others) to endeavour to discover, whether the motion of the moon's apogee, on which that objection had its whole weight and foundation, could not be truly accounted for, without supposing a change in the received law of gravitation, from the inverse ratio of the squares of the distances. The success was answerable to my hopes, and such as induced me to look farther into other parts of the theory of the moon's motion, than I first intended: but, before I had completed my design, I received the honour of a visit from M. Clairaut (just then arrived in England) of whom I learned, that he had a little before printed a piece on that subject; a copy of which I afterwards received, as a present at his hands; wherein I found most of the same things demonstrated, besides several others, to which I had not then extended my enquiry. Upon this, I at that time desisted from a farther prosecution of the subject; being chiefly diverted therefrom by a call then subsisting for a new edition of another work, in which some additions seemed wanting. But I cannot omit to observe here, in justice to M. Clairaut, that, tho' he indeed fell into a mistake, by too hastily inferring a defect in the received law of attraction, from the insufficiency of the known methods for determining the effect of that attraction, in the motion of the moon's apogee, yet he was himself, the first who discovered the true source, of that mistake, and who placed the matter in a proper light: Though there are some * who have, both before and since, undertaken to give the true quantity of that motion, from such principles, only, as are laid down in the ninth section of the first Book of the Principia: but that these Gentlemen, however they may have made their numbers to agree, have been greatly deceived in their calculations, is very certain; since a considerable part of the said motion depends on that part of the solar force acting in the direction perpendicular to the Radius-vector, which is by them, either intirely disregarded, or the effect thereof, not made one twentieth part of what it really ought to be.—There are Others indeed, who have explained the matter, upon true principles, and with better success. Since M. Clairaut's piece first made its appearance, the most eminent mathematicians, in different parts of Europe, have turned their thoughts that way. But tho' what I now offer on the same subject, may, perhaps, appear of less value, after what has been already done by these great men, yet I am not very solicitous upon that account, as it will be found, that I have neither copied from

* Vid. *Walmsley's Theorie du mouvement des apsidés* (translated into English) and Vol. 47. N^o 11. of the *Philosophical Transactions*. their

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their thoughts, nor detracted from their merit. The facility of the method I have fallen upon, will, I flatter myself, be allowed by all, who are appriz'd of the real difficulty of the subject; and the extensiveness thereof will, in some measure, appear from this, that it not only determines the motion of the apogee in the same manner, and with the same ease, as the other equations, but utterly excludes, at the same time, all terms of that dangerous species (if I may so express myself) that have hitherto embarrassed the greatest Mathematicians, and that would, after a great number of revolutions, intirely change the figure of the orbit. It thereby appears, that all the terms, or equations in general, will be expressed by sines and co-sines, barely, without any multiplication into the arcs corresponding. From whence this important consequence is derived, that the mean motion, and the greatest quantities of the several equations will remain unchanged; unless disturbed by the intervention of some foreign, or accidental cause.

In treating of this subject, as well as in most of the other parts of the ensuing work, I have chiefly adhered to the analytic method of Investigation, as being the most direct and extensive, and best adapted to these abstruse kinds of speculations. Where a geometrical demonstration could be introduced, and seemed preferable, I have given one: but, tho' a problem, sometimes, by this last method, acquires a degree of perspicuity and elegance, not easy to be arrived at any other way, yet I cannot be of the opinion of Those who affect to shew a dislike to every thing performed by means of symbols and an algebraical Process; since, so far is the Synthetic method from having the advantage in all cases, that there are innumerable enquiries into nature, as well as in abstracted science, where it cannot be at all applied, to any purpose. Sir Isaac Newton himself (who perhaps extended it as far as any man could) has even in the most simple case of the lunar orbit (Princip. B. 3. prop. 28) been obliged to call in the assistance of algebra; which he has also done, in treating of the motion of bodies in resisting mediums, and in various other places. And it appears clear to me, that, it is by a diligent cultivation of the Modern Analysis, that Foreign Mathematicians have, of late, been able to push their Researches farther, in many particulars, than Sir Isaac Newton and his Followers here, have done: tho' it must be allowed, on the other hand, that the same Neatness, and Accuracy of Demonstration, is not every-where to be found in those Authors; owing in some measure, perhaps, to too great a disregard for the Geometry of the Antients.



A

DETERMINATION
OF THE
PRECESSION OF THE EQUINOX,
And the different MOTIONS of the EARTH'S AXIS,
Arising from the ATTRACTION of the SUN and MOON.

THE PRECESSION of the *Equinox*, whereby the fix'd
stars appear to have changed their places by more
than a whole *sign*, since the time of the most ancient
Astronomers, is physically accounted for, from the at-
traction of the *sun* and *moon* on the protuberant matter about
the earth's equator; whereby the position of the said equator
with respect to the plane of the ecliptic is subjected to a per-
petual variation. Were the earth to be perfectly spherical and
of an uniform density, no change in the position of the terre-
strial equator could be produced, from the attraction of any re-
mote body; because the force of each particle of matter in the
earth, to turn the whole earth about its center, in consequence
of such attraction, would then be exactly counterbalanced by
an equal, and contrary force. But as the earth, by reason of
the centrifugal force of the parts thereof, arising from the di-
urnal rotation, must, to preserve an *equilibrium*, put on an ob-
late figure, and rise higher about the equatoreal parts than at
the poles, the action of the sun on the said equatoreal parts will
have an effect to make the plane of the terrestrial equator to
coincide with that of the ecliptic: which would actually be

B

brought

Of the Precession of the Equinox,

brought to pass (neglecting other causes) was the sun, or earth, to remain fix'd in either of the *solstices*, and the diurnal rotation at the same time to cease. But, though both the motions of the earth contribute to prevent an effect of that sort, yet, in consequence of this action of the sun, a new motion of rotation, about that diameter of the equator lying in the circle of the sun's declination, is produced; from which the precession of the equinox and the nutation of the earth's axis have their rise.

The effect of the moon, as it is much more considerable than that of the sun, so is it likewise liable to some inequalities to which that of the sun is not subject. Were the inclination of the lunar orbit to the plane of the equator to remain, always, nearly the same, like that of the earth, the same calculations that answer'd in the one case would also answer in the other; but that inclination is continually varying, and, when the ascending node is in the beginning of *Aries*, is greater by above $\frac{1}{8}$ th part than the mean value; and therefore, as the force of the moon to turn the earth about its center (other circumstances remaining the same) is found, hereafter, to be as the sine of the double of the inclination, it is manifest, that, in the said position of the node, the motion of precession will go on much quicker than at the mean rate; and consequently that an equation, depending on the place of the node, will necessarily arise. The determination of which, as well as of the other motions of precession and nutation arising from the attraction both of the sun and moon, I shall now proceed to shew: but in order to pave the way thereto, it will be proper to begin with premising the subsequent *Lemmas*.

L E M M A I.

Supposing all the particles of a given spheroid A'PáPO to be solicited parallel to the axis Pp, by forces proportional to the distances from a plane PAOPa passing by the said axis, in such sort that the two opposite semi-spheroids, A'Pp, áPp, may thereby be equally urged in contrary directions; it is proposed to determine the whole effect of all the forces to turn the spheroid about its center.

Let

Let $\begin{cases} a = \text{semi-diam. OA'} \text{ (perpend. to the plane PAOpa), } & \text{Fig. 1.} \\ A = \text{area of the ellipse PAOpa,} \\ \gamma = \text{force acting on a particle at the remotest point A',} \\ x = \text{ON, the dist. of any section DQNEQ from PAOpa:} \end{cases}$

Then, this section being also an ellipse, similar to PApa, we shall have, by the property of the ellipsis, as $A'O^2(aa) : A'O^2 - ON^2(aa - xx) :: PO^2 : DN^2 :: \text{the area PAOpa (A) to the area DQNEQ} = A \times \frac{aa - xx}{aa}$ (by the property of similar

figures). Hence it is evident that $A \times \frac{aa - xx}{aa} \times \frac{x}{a} \times \gamma$ will be the sum of all the forces whereby the particles in the ellipse DQEQ are urged parallel to the axis Pp of the spheroid; which quantity, drawn into (x) the length of the lever ON, will, consequently, express the effect of all the said forces to turn the spheroid about its center: and so the fluent of $A \times \frac{aa - xx}{aa} \times \frac{x}{a} \times xx \times \gamma$, which is $A \times \frac{2aa}{15} \times \gamma$ (when $x = a$) will truly express one half of the quantity sought.

C O R O L L A R Y.

If the mass, or content of the spheroid, which is $A \times 2a \times \frac{2}{3}$, be denoted by S ; then the force $A \times \frac{4aa}{15} \times \gamma$, whereby the spheroid tends to turn about its center, will be truly defined by $\frac{4}{15} S \times a \times \gamma$, which therefore is just $\frac{4}{15}$ th part of what it would be, if all the particles were to act at the distance of the remotest point A'.

L E M M A II.

Suppose a body to revolve in the circumference of a circle AFaF, Fig. 2. whilst the circle itself turns uniformly about one of its diameters Aa, as an axis, with a very slow motion; it is proposed to determine the law of the force, acting on the body in a direction perpendicular to the plane of the circle, necessary to the continuation of a motion thus compounded.

Fig. 2. Let $AFaF$ and $Afaf$ be two positions of the circle, indefinitely near to each other, and let R and r be the two corresponding positions of the body; let also the planes RDn and mdc be perpendicular to $AFaF$ and to the axis AOa ; in which planes let there be drawn Rn and mv perpendicular to DR and dm , meeting the plane $Afaf$ (produced out) in n and c ; and let there be drawn rv , parallel to the tangent Rtm , meeting mc in v . If the velocity of the body along the circumference be expressed by Rm , the velocity in the perpendicular direction Rn , arising from the motion of the circle about the axis Aa , will be represented by Rn . And, if the body were to be suffered to pursue its own direction from the point R , it would, by the composition of those motions, arrive at the opposite angle v of the parallelogram $Rrvv$, in the same time that it might move through Rm by the motion Rm alone; and so would fall short of the plane by the distance cv *. It therefore appears that the required force, necessary to keep the body in the plane, must be such as is sufficient to cause a body to move over the distance cv in the aforesaid time; and that *this force* must, therefore, be to the centrifugal force of the body in the circumference (whose measure is et) as cv to et ; since the spaces described in equal times, are directly as the accelerating forces.

Let now the ratio of the angular celerity of the circle about its axis to *that* of the body in the circumference, be supposed as r to unity; then, the latter of these celerities being represented by Rm , the former will be defined by $r \times Rm$; and consequently the celerity (Rn) in the direction Rn , by $r \times Rm \times \frac{DR}{OF}$. Moreover, because of the similitude of the triangles DRn and dmc , it will be, as $DR : Rn$ ($r \times Rm \times \frac{DR}{OF}$) :: dm ($DR + sm$) : $mc = r \times \frac{Rm \times DR}{OF} + r \times \frac{Rm \times sm}{OF}$: from whence, taking away the value of mv or Rn , we get $cv = r \times \frac{Rm \times sm}{OF}$: which is

* The lineola cr , lying in the plane of the circle, must be answered by a force tending to the center of the circle; with which we have nothing to do in the present consideration.

in proportion to the measure of the centrifugal force et , or it's equal $\frac{Rm^2}{2OR}$, as $r \times sm$ to $\frac{1}{2}Rm$, or, because of the similar triangles ORD and Rsm, as $2r \times OD$ to OR or OA.

Hence it is evident that the body, to continue in the plane of the circle, must be *constantly* acted on, in a direction perpendicular to the plane, by a force varying according to the cosine of the distance AR of the body from the extremity of the axis; whose greatest value, at A, is to the centrifugal force in the circle, as $2r$ to unity. Q. E. I.

COROLLARY I.

If, instead of one, a great number of bodies or corpuscles, so as to touch one another and thereby form a continued ring AFaF, were to revolve at the same time, and to be acted on in the same manner (that is to say, by forces in the ratio of the distances from the diameter FF perpendicular to the axis Aa), it is evident that they would all continue in the same plane. And this will also be the case, when a number of concentric rings ERGeG, &c. are supposed to perform their revolutions together about the common axis AEea. For, assuming β to denote the centrifugal force of a corpuscle in the outermost ring AR'FaF, the centrifugal force of an equal corpuscle (R') in the ring ER'eG, will be equal to $\beta \times \frac{OE}{OA}$: whence, by the foregoing proportions, $2r \times \beta \times \frac{OE}{OA}$ will be the force acting perpendicular to the plane at E: and $2r \times \beta \times \frac{OE}{OA} \times \frac{OD}{OR}$ ($= 2r \times \beta \times \frac{OD}{OA}$) will be the true measure of the force acting on a corpuscle at R'; which, as r , β , and OA are all of them constant, is evidently as the distance from the diameter FF. Whence it follows, because the distance below FF becomes negative, that the forces above and below that diameter must have contrary directions.

COROLLARY II.

Whatever hath been said in the preceding Corollary holds equally,

Fig. 4.

equally, when the line or axis Aa , about which the plane is supposed to turn, hath a progressive motion, or is carried uniformly forward, parallel to itself; provided the angular celerity about that axis continues the same; as is evident from the resolution of forces. Hence it follows, that, if a circle $E'ECe$, considered as composed of an indefinite number of concentric rings, be supposed to revolve uniformly about its center C , whilst the center itself and the right-line OC (which, to help the imagination, may be taken as the axis of a cone $E'Oe$, whose base is $E'Ee$) move uniformly in the plane $P\acute{a}pA'$ about the point O ; I say, it follows that the forces necessary to keep the particles in the plane, under such a compound motion, will be the very same as if the circle was to turn about the line Ee (perpendicular to the plane $P\acute{a}pA'$) at rest, with an angular celerity equal to *that* of the center C about the point O : because, the angle OCE' being *always* a right one, the angular celerity of the moveable circle about the line ECe (which remains every-where parallel to itself) will, evidently, be equal to the angular celerity of the center of the circle about the point O . From whence and the preceding Corollary it is manifest, that the forces which, acting parallel to PCO , are necessary to retain the particles in the plane $E'Ee$, will be, every-where, as the distances from the diameter $E'Ce$, or the plane $P\acute{a}pA'$, let the distance of the plane $E'Ee$ from the center O be what it will.

COROLLARY III.

Conceive now $OAP\acute{a}pA'd$ to be an homogenous fluid, revolving uniformly about the axis POp , under the form of an oblate spheroid*; whilst the axis itself is supposed to turn about the center O , in the manner explained above: then it will appear, from what is there delivered, that the particles of the fluid, to continue in *equilibrio* among themselves, must be solicited parallel to the axis, by forces that are as the distances from the plane $P\acute{a}pA'$; such, that the force acting at the remotest point A may be defined by $2r\beta$; where β (by *Corol. I.*)

* That the particles will remain in equilibrio, under the form of an oblate spheroid (when the axis is at rest), is demonstrated in Part II. Sect. 9. of my *Doctrine and Application of Fluxions*.

represents

represents the centrifugal force in the circumference $AaaA'$ of the greatest circle, and r the measure of the angular motion of the axis itself, that of the rotation, about the axis, being denoted by *unity*. But it appears further, from *Lemma I*, that the efficacy of all the said forces to turn the spheroid about its center (making γ here $= 2r\beta$) is truly defined by $2r\beta \times \frac{1}{2}S \times OA$. Whence it is plain, that all the particles of the body will remain *in equilibrio* among themselves, under the two different motions above explained, when the *whole* force producing the motion of the axis, is expressed by $2r\beta \times \frac{1}{2}S \times OA$. And, when the forces respecting the several particles are supposed to act according to a different law, the effect produced by them will be the same, provided their *joint efficacy*, to turn the body about its center, *be the same*: since the same force must be answered, or satisfied with the same kind and degree of motion in the whole body; if we except only, the exceeding small difference that will arise from the alteration of the figure; which figure will not be *accurately* a spheroid, in this case, but *nearly* such, as the motion of the axis and, consequently, the forces producing it, are supposed very small. Neither will the axis continue to move in the same plane, when the direction of the forces is not every-where parallel to the axis; the motion produced in the body being always about that diameter (Aa) wherein the whole perturbing force may be conceived to act, as by a lever, to turn the body about its center. Lastly, it may be observed here, that the time of revolution about the axis will not, in this case, continue *accurately* the same; since a change of the figure must necessarily be attended with a change in the time of revolution. But this change of motion about the axis, when we regard the effect of the perturbing forces of the sun and moon upon the earth, is so extremely small, as to be quite inconsiderable, even in comparison of the very slow motion of the axis above spoken of.

Fig. 4.

LEMMA III.

Supposing all the particles of a given ellipse $MFNf$ to be urged from a right-line GG coinciding with a given diameter MN , by forces

Of the Precession of the Equinox,

forces proportional to the distances from the said line, such that the force acting at a given distance a , may be expressed by a given quantity γ ; it is required to find the whole efficacy of all these forces, to turn the ellipse about its center O .

If BC be supposed parallel to GG , intersecting OT , perpendicular thereto, in D ; then the force with which a particle, at any place V in that line, is urged in the direction wV parallel to OD , will be expressed by $\frac{\gamma}{a} \times Vw$, or $\frac{\gamma}{a} \times OD$; and consequently it's efficacy to turn the ellipse about its center by $\frac{\gamma}{a} \times OD \times Ow$, or $\frac{\gamma}{a} \times OD \times DV$. Let there be taken $Cv = DV$; and the efficacy of a particle at v will, in like manner, be had equal to $\frac{\gamma}{a} \times OD \times Dv$: which, added to that of the

Fig. 5. former particle at V , gives $\frac{\gamma}{a} \times OD \times DC$. Therefore, seeing the joint action of any two particles in DC , equally distant from the middle one I , is expressed by the same quantity $\frac{\gamma}{a} \times OD \times DC$, the efficacy of all the particles must consequently be equal to that quantity drawn into half the number of the particles; and so is truly expounded by $\frac{\gamma}{a} \times \frac{1}{2} OD \times DC^2$. By the same argument, the force of all the particles in the line BD to turn the ellipse about its center, the contrary way, will be $\frac{\gamma}{a} \times \frac{1}{2} OD \times BD^2$. Therefore the difference of these two values, $\frac{\gamma}{a} \times \frac{1}{2} OD \times \overline{BD^2 - CD^2}$, is the whole force of all the particles in the line BC , to turn the ellipse about its center (downwards); which expression, if Ff the conjugate diameter to MN be drawn, bisecting BC in E , will become $\frac{\gamma}{a} \times \frac{1}{2} OD \times \overline{BD + CD} \times \overline{BD - CD} = \frac{\gamma}{a} \times OD \times BC \times DE$.

Put, now, $OF = c$, $OM = d$, FH (perpendicular to MN) $= f$, $OH = g$; and let OE and OD , considered as variable, be denoted by x and y , respectively. Then, by the property of the

the ellipsis, it will be, $cc : dd :: cc - xx : \overline{BE}^2 = \frac{dd \times cc - xx}{cc}$;

and consequently $BC = \frac{2d\sqrt{cc-xx}}{c}$. Also (by similar trian-

gles) $c : f :: x : y = \frac{fx}{c}$; and $c : g :: x : DE = \frac{gx}{c}$. Hence

our expression $\frac{\gamma}{a} \times OD \times DE \times BC$, derived above, by substitut-

ing these values, becomes $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc-xx}^{\frac{1}{2}} \times x^2$: and there-

fore the whole fluent of $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc-xx}^{\frac{1}{2}} \times xxj$, or of its equal

$\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc-xx}^{\frac{1}{2}} \times x^2 \times \frac{fx}{c}$, will be the force of all the parti-

cles in the semi-ellipse MFN. In order to the finding of this

fluent, let A be taken to denote the area of the semi-ellipse,

or, which is the same, the fluent of $\frac{2d}{c} \times \overline{cc-xx}^{\frac{1}{2}} \times \frac{fx}{c}$; then,

by comparison, the whole fluent of $\frac{2d}{c} \times \overline{cc-xx}^{\frac{1}{2}} \times x^2 \times \frac{fx}{c}$, when

$x=c$, will be found to be $A \times \frac{1}{4}c^2$: whence that of our given

expression, $\frac{\gamma}{a} \times \frac{2dfg}{c^3} \times \overline{cc-xx}^{\frac{1}{2}} \times x^2 \times \frac{fx}{c}$, must consequently be

$= \frac{\gamma}{a} \times \frac{fg}{cc} \times \frac{1}{4}c^2 A = \frac{\gamma}{a} \times \frac{1}{4}fgA = \frac{\gamma}{a} \times \frac{1}{4}FH \times OH \times A$; the

double of which, or $\frac{\gamma}{a} \times \frac{1}{4}FH \times OH \times \text{area MFNfM}$, is there-

fore the true measure of the whole force whereby the ellipse

tends to move about its center. Q. E. I.

COROLLARY I.

If the same value be required by means of the angle AOM included between the diameter MN and the principal axis AOa (supposed to be given); then let POp and FR be drawn perpendicular to OA, and TF to OT, meeting OA produced, in Q; suppose L to be the intersection of AO and FH; and let Fig. 6.
the sine and co-sine of the said given angle AOM (to the radius 1) be denoted by m and n , respectively. Because FL is perpendicular to the tangent TQ, we have, by the property of
C the

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the ellipsis, as $AO^2 : AO^2 - OP^2 :: OR : OL :: OR \times OQ$
 $(AO^2) : OL \times OQ$; and consequently $AO^2 - OP^2 = OL \times OQ$.

But, $1 : n :: OL : OH$;

and, $1 : m :: OQ : OT (FH)$;

whence, by composition, $1 : mn :: OL \times OQ (= AO^2 - OP^2)$
 $: FH \times OH = mn \times AO^2 - OP^2$: and so, by substituting this
 value above, we get $\frac{\gamma}{a} \times \frac{mn}{4} \times AO^2 - OP^2 \times \text{area of the ellipse}$,
 for another expression of the required force.

COROLLARY II.

Hence may be easily deduced the force by which the spheroid, generated by the rotation of the ellipse about its lesser axis Pp , tends to turn about its center, when all the particles are urged from a plane GG passing through the center, by forces proportional to the distances from the said plane. For, as any section of the spheroid, parallel to the middle one $ApaP$, is also an ellipse, similar to it, the area of that section will be in proportion to the area of $ApaP$ (which I shall denote by $\mathcal{Q}$) as the square of its greater semi-axis, to the square of the greater semi-axis OA of the given ellipse $Papa$: so that, if OA be denoted by a , PO by b , and the distance of the said section from the center of the spheroid by u , we shall have,

$aa : aa - uu (= \text{sq. greater semi-axis of that section, by the property of the circle}) :: \mathcal{Q} : \mathcal{Q} \times \frac{aa - uu}{aa}$, the area of the section. Moreover, by reason of the similar figures, we have

$aa : aa - bb :: aa - uu : \frac{aa - bb}{aa} \times aa - uu$, the difference of the squares of the greater and lesser semi-axes of the section.

Therefore, by substituting these values in the above general expression, we get $\frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa - bb}{aa} \times \frac{aa - uu}{aa} \times \mathcal{Q} \times \frac{aa - uu}{aa}$

$(= \frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa - bb}{aa} \times \mathcal{Q} \times \frac{a^4 - 2a^2u^2 + u^4}{aa})$ for the force of all the particles in that section to turn the body about the common axis of motion standing at right-angles to the plane $Papa$. This quantity, drawn into $\dot{u}$, will, therefore, be the fluxion of force of the
 semi-

semi-spheroid in which that section is; whose fluent, when $u=a$, will be found $= \frac{\gamma}{a} \times \frac{mn}{4} \times \frac{aa-bb}{aa} \times Q \times \frac{8a^3}{15}$: the double whereof, or $\frac{\gamma}{a} \times mn \times \frac{aa-bb}{4} \times \frac{4aQ}{15}$, must consequently be the required force of the whole spheroid: which force, as $Q \times \frac{4a}{3}$ is known to express the content, or mass of the spheroid, will also be truly defined by $\frac{\gamma}{a} \times \frac{mn}{5} \times \frac{aa-bb}{5} \times S$; S being put (as in the preceding Lemmas) to represent the said content or mass.

PROBLEM I.

To determine the efficacy of the sun's attraction, on a corpuscle, any where in the body of the earth, to turn the earth about it's center.

Let CDHE represent the earth, C the center thereof, S that Fig. 7. of the sun, and GCG a plane perpendicular to the line CS joining the centers of the earth and sun; let D be the place of the corpuscle, and upon the diagonal SD let the parallelogram QCSD be constituted; producing QD to meet GCG in K.

If F be taken to denote the sun's absolute force on a particle at the center C, his force on a particle at D will be $F \times \frac{SC^2}{SD^2}$; which may be resolved into two others, the one in the direction DC (which has no effect at all to turn the earth about its center); and the other in the direction DQ, expressed by $F \times \frac{SC^2}{SD^2} \times \frac{DQ}{SD}$: from which the force F , in the parallel direction CS, being deducted, the remainder $F \times \frac{SC^2 - SD^2}{SD^3}$ will be the true measure of that part of the force in the direction DQ, whereby the particle at D tends to change its position with respect to the plane GCG. But this value is reducible to $F \times \frac{SC - SD \times \frac{SC^2 + SC \times SD + SD^2}{SD^3}}{SD^3}$; which, as $SC - SD$ (by reason of the great distance of the sun) is nearly equal to

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DK, will become $= F \times \frac{DK \times 3SD^2}{SD^3} = 3F \times \frac{DK}{SC}$, nearly :
 which, drawn into CK, will be as the required efficacy of
 that force to turn the body about its center. *Q. E. I.*

COROLLARY I.

If there be $\left\{ \begin{array}{l} A = (SC) \text{ the distance of the sun and earth,} \\ a = \text{the semi-equatoreal diameter of the earth,} \\ T = \text{the time of the annual revolution,} \\ t = \text{the time of the diurnal revolution, and} \\ \beta = \text{the centrifugal force of a particle at the equator, arising from the diurnal revolution;} \end{array} \right.$
 taken

then, since $\frac{a}{t^2} : \frac{A}{T^2} :: \beta : F$, or $F = \frac{A\beta t^2}{aT^2}$ (by the known laws

of central forces) it is evident that the force $3F \times \frac{DK}{SC}$, with
 which a particle at D tends from the plane GCG, will also be
 truly defined by $\beta \times \frac{3t^2}{T^2} \times \frac{DK}{a}$. Hence it appears that the said
 force is directly as the distance from the plane; and that the
 value thereof, at the distance of the earth's semi-equatoreal
 diameter, is truly defined by $\beta \times \frac{3t^2}{T^2}$; being in proportion to the
 centrifugal force at the equator, arising from the diurnal rota-
 tion, as thrice the square of the time of the diurnal revolution,
 to once the square of the time of the annual revolution.

COROLLARY II.

Fig. 7. Moreover, from hence the perturbing force of the moon,
 or any other planet, will be given: for, supposing S to repre-
 sent the planet, it's absolute force (F) at the center C, will be
 as its quantity of matter, applied to the square of the distance
 SC: and so our general expression $3F \times \frac{DK}{CS}$ will here become
 $\frac{3DK \times S}{SC^2}$; which, because the quantities of matter in bodies
 of the same density, are as the cubes of their semi-diameters,
 will also (supposing the position of D to remain the same) be

be as the cube of the semi-diameter of the planet directly, and the cube of its distance SC, inverfly; or, which is the same, as the cube of the sine of the apparent semi-diameter directly, and the cube of the radius inverfly. Hence it is manifest, that the perturbing forces of planets, of the same density, are in proportion directly as the cubes of the sines of their apparent semi-diameters, or as the cubes of the semi-diameters, themselves, very near. Therefore the sun and moon, appearing under equal semi-diameters, have their perturbing forces in the same ratio with their densities.

PROBLEM II.

To determine the change of the position of the terrestrial equator, arising from the action of the sun on the whole mass of the earth, during any very small interval of time.

Let OAPap be the earth, under the form of an oblate Fig. 8. spheroid; let AIAI be the plane of its equator, and HICL a plane passing through S the center of the sun, and making right-angles with the plane of the meridian HAPCp.

It appears, by *Corol. I. to Prop. I.* that the relative force whereby a particle of matter, any where in the earth, tends, through the action of the sun, to recede from the plane GG, perpendicular to HICL, is directly as the distance from the said plane; and that the said force, at the distance (a) of the semi-equatoreal diameter from the plane, is truly defined by $\beta \times \frac{3tt}{T^2}$.

It appears moreover, from *Corol. II. to Lem. III.* that a spheroid, acted on in this manner, tends, through the joint force of all the particles, to turn about its center with a force expressed by $\frac{\gamma}{a} \times \frac{mn \times aa - bb}{5} \times S$; γ being the force wherewith a particle, at the distance a , is urged from the plane GG. Therefore, expounding γ by $\beta \times \frac{3tt}{T^2}$, we have $\beta \times \frac{3tt}{T^2} \times \frac{mn \times aa - bb}{5a} \times S$, for the true measure of the force whereby the whole earth tends to turn about its center, through the sun's attraction.

But

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But it is proved, in *Corol. III. to Lem. II.* that, if a spheroid *OAPap* revolving about its axis *Pp*, be at the same time acted on by forces tending to generate a new motion, at right angles to the former, it will, in consequence of such action, have another motion, about the line *Aa*; whereof the celerity will be in proportion to *that* of the former motion about the axis (*Pp*), as *r* to 1; the whole force with which the spheroid tends to turn about its center, whereby this motion is produced, being expressed by $2r\beta \times \frac{r}{5} S \times a$. Let this force, therefore, be made equal to *that* found above, by which the earth tends to turn about its center: by which means we have

$$\frac{2ra}{5} = \frac{3tt}{2T} \times \frac{mn \times \overline{aa - bb}}{5a}; \text{ and therefore } r = \frac{3tt}{2T} \times \frac{mn \times \overline{aa - bb}}{aa}.$$

From whence it appears that the earth, in consequence of the sun's attraction, has a motion about the line *Aa* (lying in the plane of the sun's declination) whereof the celerity will be in proportion to *that* of the diurnal motion about the axis *Pp*, as

$$\frac{3tt}{2T} \times \frac{mn \times \overline{aa - bb}}{aa} \text{ to unity: where } t \text{ and } T \text{ express the respective times of the diurnal, and annual revolutions, } a \text{ and } b \text{ the greatest, and least semi-diameters of the earth, and } m \text{ and } n \text{ the sines of the sun's declination and polar distance.}$$

PROBLEM III.

To determine the precession of the equinox, and the nutation of the earth's axis, caused by the sun, during any very small interval of time; on the supposition of an uniform density of all the parts of the earth.

Fig. 9. Let $\triangle D\varphi$ be the ecliptic, on the surface of the sphere; and let $\triangle AC\varphi$ be the position of the equator, when *S* is the sun's place in the ecliptic, and *SA* his declination.

It is evident, from the last proposition, that the angle $\triangle Aa$, or φAb , described by the equator about the point *A*, in any very small time *t*, by means of the sun's attraction, is in proportion to $\left(\frac{t}{T} \times 360^\circ\right)$ the angle described in the same time, by

means

means of the diurnal motion, as $\frac{3tt'}{2TT} \times \frac{mn \times \overline{aa-bb}}{aa}$ is to unity; and therefore is truly defined by $360^\circ \times \frac{3tt'}{2TT} \times \frac{mn \times \overline{aa-bb}}{aa}$, or $360^\circ \times \frac{3tt'}{2TT} \times kmn$; supposing $\frac{aa-bb}{aa}$ (for the sake of brevity) to be denoted by k . But it will be (*p. spherics*) as sine of a (or $\sphericalangle$) : sin. $\sphericalangle A$ (:: sin. $\sphericalangle Aa$: sin. $\sphericalangle a$) :: angle $\sphericalangle Aa$: $\sphericalangle a = 360^\circ \times \frac{3tt'}{2TT} \times kmn \times \frac{\text{fin. } A}{\text{fin. } \sphericalangle}$, the quantity of the precession required.

Again, if aD and ac be taken as arcs of 90 degrees each; so that Dc may be the measure of the angle a ; we shall also have (*p. spherics*) as Rad. : sin. AC (co-fin. $\sphericalangle A$) :: sin. CAC ($\sphericalangle Aa$) : sin. Cc :: angle $\sphericalangle Aa$: $Cc = 360^\circ \times \frac{3tt'}{2TT} \times kmn \times \frac{\text{co-f. } \sphericalangle A}{\text{Rad.}}$, the required nutation, or the decrease of the inclination of the equator to the ecliptic. *Q. E. I.*

COROLLARY.

If $\sphericalangle e$ be made perpendicular to aA , it will be, as $\sphericalangle e$: Cc (:: sin. $\sphericalangle A$: sin. CA) :: tang. $\sphericalangle A$: Radius;

also, $\sphericalangle a$: $\sphericalangle e$:: Rad. : sin. a : whence, by compounding these proportions, we have $\sphericalangle a$: Cc :: tang. $\sphericalangle A$: sin. a . From which it appears, that the quantity of the precession, for any very small time, is to that of the nutation corresponding, as the tangent of the sun's right ascension to the sine of the inclination of the equator to the ecliptic.

PROBLEM IV.

To determine the precession of the equinox, and the nutation of the earth's axis, caused by the sun, from the time of his appearing in the equinoctial point, to his arrival at any given distance therefrom.

Every thing being supposed as in the preceding Problem, put Fig. 9. the sine of the angle $\sphericalangle = p$, its co-sine $= q$, the arch $\sphericalangle S = x$,
its

its sine $= x$, its cosine $= y$, and the length of the semi-periphery $\simeq D\varphi = e$: then, *p. spherics*, it will be, co-fin. $AS \times \text{co-fin.} \simeq A (= y \times \text{Rad.}) = y$; and fin. $AS (= \frac{px}{\text{Rad.}}) = px$: whence, by multiplying these two equations together, we have fin. $AS \times \text{co-fin.} AS \times \text{co-fin.} \simeq A (= mn \times \text{co-fin.} \simeq A) = pxy$: and so, by substituting this value for its equal, our expression for the nutation, during the time t' (given by the last Problem) will here be reduced to $360^\circ \times \frac{3t'}{2T} \times kpxy$.

But the time wherein the sun's longitude $\simeq S$ is augmented by the particle $\dot{z}$ will be $T \times \frac{\dot{z}}{2e}$; which being wrote in the room of t' , we thence have $360^\circ \times \frac{3t}{4T} \times \frac{kp}{e} \times xy\dot{z}$: and *this*, by substituting $\sqrt{1 - xx}$, and $\frac{\dot{x}}{\sqrt{1 - xx}}$, instead of their equals y and $\dot{z}$, will be farther transformed to $360^\circ \times \frac{3t}{4T} \times \frac{kp}{e} \times x\dot{x}$. Whose fluent, $360^\circ \times \frac{3t}{4T} \times \frac{kp}{2e} \times xx$, is consequently the true quantity of the nutation that was to be determined.

Again, with regard to the precession of the equinox, the increase thereof (*by the Corol. to the precedent*) being in proportion to the corresponding decrement of the inclination of the equator to the ecliptic, as the tangent of $\simeq A$ to the sine of $\simeq$, or, in species, as $\frac{qx}{\sqrt{1 - xx}}$ to $\dot{p}$, it therefore appears (by multiplying the fluxion of the nutation by $\frac{qx}{p\sqrt{1 - xx}}$) that $360^\circ \times \frac{3t}{4T} \times \frac{kq}{e} \times \frac{x^2\dot{x}}{\sqrt{1 - xx}}$ will be the fluxion of the quantity under consideration; whose fluent, which is $360^\circ \times \frac{3t}{4T} \times \frac{kq}{e} \times \frac{x - x\sqrt{1 - xx}}{2}$, is therefore the precession itself. *Q. E. I.*

COROL.

COROLLARY I.

When the sun arrives at the solstitial point D, the value of x being $= 1$, and that of $z = \frac{1}{2}e$, the quantity of the precession becomes barely equal to $360^\circ \times \frac{3t}{4T} \times \frac{kq}{4}$; whose quadruple, $360^\circ \times \frac{3t}{4T} \times kq$, will be the whole of the annual precession, depending on the sun; which, in numbers (by making $t = 1$, $T = 366\frac{1}{4}$, $q = .91723 = \text{co-sine of } 23^\circ 28' \frac{1}{2}$, $a = 231$, $b = 230$, and $k (= \frac{aa-bb}{aa}) = \frac{2}{230\frac{1}{2}}$) comes out $21'' 7'''$. But it will appear from what follows hereafter, that this quantity, derived on the hypothesis of an uniform density of all the parts of the earth, ought to be reduced to about $14\frac{1}{2}''$, to agree with observations.

COROLLARY II.

Since the precession during $\frac{1}{4}$ th of the annual revolution is found to be $360^\circ \times \frac{3t}{4T} \times \frac{kq}{4}$, we have as $\frac{1}{2}e : z :: 360^\circ \times \frac{3t}{4T} \times \frac{kq}{4} : (360^\circ \times \frac{3t}{4T} \times \frac{kq}{2e} \times z)$ the mean precession during the time of describing the arch: which being taken from the true precession, $360^\circ \times \frac{3t}{4T} \times \frac{kq}{2e} \times z - x\sqrt{1-xx}$, the remainder, $360^\circ \times \frac{3t}{4T} \times \frac{kq}{2e} \times x - x\sqrt{1-xx}$, will consequently be the *equation* of the precession; which therefore is to the mean precession, as $-x\sqrt{1-xx} : z$, or as $-2x\sqrt{1-xx} : 2z$; that is, as $-\sin. 2z : 2z$. But the mean precession, in the time of describing the arch z , is $21'' 7''' \times \frac{z}{2e}$ (or rather $14\frac{1}{2}'' \times \frac{z}{2e}$) by *Corol. I.* Therefore the equation corresponding will be $-\frac{21'' 7'''}{4e} \times \sin. 2z = -\frac{1'' 42'''}{1} \times \sin. 2z$, when the density is taken as uniform; but when taken to correspond with the observation, it will be $-\frac{14\frac{1}{2}''}{4e} \times \sin. 2z = -1'' 10'' \times \sin. 2z$. Hence

D

it

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it appears, that the greatest equation of the precession (when the sun is in the mid-way between the equinox and solstice) is $1'' 10'''$; and that the general equation (which is subtractive in the first and third quadrants of the ecliptic) will be in proportion to the said greatest equation, as the sine of twice the sun's distance from the equinoctial point is to the radius.

COROLLARY III.

Furthermore, because the quantity of the nutation is, universally, equal to $360^\circ \times \frac{3t}{4T} \times \frac{kpx^2}{2e}$, it will therefore be the greatest possible, when the sun is in the solstice and x is the greatest possible: after which it will decrease, according to the same law whereby it before increased; 'till, on the sun's arrival at the other equinoctial point, it intirely vanishes, and the inclination is thereby restored to its first quantity. It is also evident that the quantity of the nutation will, in all circumstances, be in proportion to the precession, during $\frac{1}{4}$ th of the sun's revolution, as $\frac{px^2}{e}$ to $\frac{q}{2}$, or as $\frac{px^2}{q}$ to $\frac{e}{2}$, that is, as the product under the square of the sine of the sun's longitude and the tangent of the inclination of the two planes of the equator and ecliptic, is to the length of an arch of 90 degrees. According to which proportion (taking the said precession = $\frac{1}{4}$ th of $14\frac{1}{2}''$) the greatest nutation comes out one *second*, very near; the inclination of the two planes decreasing from the time of the sun's leaving the equinoctial points, to his arrival at the solstices, and that in the duplicate ratio of the sine of his distance from the said equinoctial points.

It may be observed that, in order to avoid trouble, the quantities p and q are taken as constant; the error, or difference thence arising scarcely amounting to $\frac{1}{100000}$ th part of the whole value.

SCHOLIUM.

Sir ISAAC NEWTON, in finding the precession of the equinox, considers the protuberant matter about the earth's equator, as a ring of moons, revolving uniformly round the center of the

the earth in 24 hours; and by virtue of that assumption, from the motion of the lunar nodes, before determined, he infers the motion of the nodes of the said ring, and from thence the precession of the equinox. We have proceeded upon other principles, and by a very different method; and it may be worth while to remark here, that, as the precession of the equinox is deducible from the motion of the nodes of a satellite, so, on the contrary, the motion of the nodes of a satellite may be very easily deduced, as a Corollary, from our general formula, for the precession of the equinox.

Thus if the value of b be supposed indefinitely small (so that the figure of the earth, or spheroid, may be conceived as flat as possible) we shall have $k (= \frac{aa - bb}{aa}) = 1$; and the ex-

pression in Corol. I. will then become $360^\circ \times \frac{3t}{4T} \times q$, exhibiting the motion of the node of a ring, or of a number of concentric rings, during the time (T) of one whole revolution of the body about the sun (*vid. Corol. I. to Lem. II.*) But it will also appear, from the articles here referred to, that the place of a satellite, moving in a circular orbit, will always be found in a ring or plane revolving in the manner there specified.

Hence $360^\circ \times \frac{3t}{4T} \times q$ will likewise express the motion of the node of a single moon, or satellite, in the time (T) of one whole revolution of the primary planet: which value, when the inclination to the ecliptic is but small, will be equal to $360^\circ \times \frac{3t}{4T}$, nearly.

Hence the mean motion of the node of a satellite, in a circular orbit, is in proportion to the mean motion of the primary planet about the sun, as $\frac{3}{4}$ ths of the periodic time of the satellite, is to the whole periodic time of the primary planet. It follows moreover, that, let a planet have ever so many satellites, the mean motions of the nodes of them all will be in proportion, directly as the times of revolution of the satellites themselves; and, consequently, the periodic times of the nodes, inversely, as the periodic times of the respective satellites.

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The proportions used by Sir ISAAC NEWTON, in inferring the precession of the equinox from the motion of the lunar node, agree exactly with those above determined; it may, therefore, seem the more strange that there should be so wide a difference between the conclusion derived, in Corol. I. of the last Problem, and that brought out by that celebrated Author; who makes the quantity of the annual precession, depending on the sun, to be no more than $9'' 7''' 20^{\sim}$: which is not the half of what it is here found to be. But to give the Reader what satisfaction I can in this particular, and to discover the error (if any such should have crept into my calculations) I shall now attempt the solution by a different method: in order to which it will be requisite to premise the two following Lemmas.

L E M M A IV.

If every particle in the circumference AIA'L of a given circle tends to turn the circle about one of its diameters IL, as an axis, by a force proportional to the square of the distance therefrom; it is proposed to find the whole force of all the particles, to turn the circle about that diameter.

Fig. 10. If, from any point E in the circumference, there be drawn EH perpendicular to the given diameter IL, the force of a particle at E will, by hypothesis, be defined by EH^2 ; which quantity, drawn into (Em) the fluxion of the arch IE, will therefore be the fluxion of the quantity to be determined. But Em , because of the similar triangles Emn , EOH , will be equal to $\frac{OE \times En}{EH}$; and therefore the fluxion sought $= OA \times EH \times En$. But $EH \times En$ is the fluxion of the area IHE: whence it is evident, that the force of all the particles in the whole circumference, will be truly defined by $OA \times \text{area of the circle}$, or $OA \times OA \times \text{the semi-circumference}$, or $OA^2 \times \frac{1}{2} \times \text{the number of particles}$. Q. E. I.

C O R O L L A R Y.

Since the force of a particle at A is expressed by OA^2 , it follows that the force of all the particles in the whole circumference will be equal to half the force of an equal number of particles acting at the distance of the highest point A.

L E M M A

LEMMA V.

To determine the momentum of rotation of a given spheroid APaOp, revolving uniformly about its axis Pp, with a given angular celerity.

Let ENF be an ordinate to the generating ellipsis APap, parallel to the axis of rotation Pp: make AO (perpendicular to Pp) = a ; OP (= Op) = b ; ON = x ; EN = y ; and let p denote the semi-periphery of the circle whose radius is unity.

Then it will be, as $1 : 2p :: x : 2px$, the periphery of the circle generated by the point N. Therefore $2px \times 2y$ will be the measure of the surface generated by the ordinate EF, in the revolution of the ellipsis about its axis Pp: which, drawn into the square of the distance ON, gives $4pyx^3$ for the momentum of rotation of all the particles in the said surface: so that the fluent of $-4pyx^3\dot{x}$ will be the true measure of the force to be determined.

Fig. 11.

Now, by the property of the ellipsis, we have $x^2 = \frac{aa}{bb} \times \overline{bb - yy}$; and consequently $x\dot{x} = \frac{-aay\dot{y}}{bb}$: whence, by substituting $\frac{a^2}{b^2} \times \overline{b^2 y\dot{y} - y^3\dot{y}}$ in the room of its equal $-x^3\dot{x}$, our fluxion is transformed to $\frac{4pa^4}{b^4} \times \overline{b^2 y^2\dot{y} - y^4\dot{y}}$: whose fluent, when $y = b$, is found equal to $\frac{8pa^4b}{15}$.

COROLLARY.

Since $\frac{4pa^2b}{3}$ is known to express the mass or content of the spheroid, the momentum of rotation of any spheroid about its axis appears, therefore, to be just the same as would arise, if $\frac{2}{3}$ ths of the whole mass was to revolve at the distance of the highest point (A) from the axis of motion.

PROBLEM V.

To determine the alteration of the position of the terrestrial equator, arising from the action of the sun on the whole mass of the earth, during an instant of time.

Let

Fig. 12.

Let $OAPap$ be the earth, under the form of an oblate spheroid; let $AlaL$ be the plane of its equator, and $HICL$ a plane passing thro' S the center of the sun, making right-angles with the plane of the meridian $HAPCp$ and with the plane GG . It is found, in *Prob. I.* that the force whereby a particle, at any point E in the equator $AELaI$, tends from the plane GG , is in proportion to that respecting the highest point A , as the distance EF to the distance AK , or as ED to AO (supposing EF parallel to AK , and ED to AO); whence it is evident that the force on the particle at E , in a direction perpendicular to the plane of the equator, must be to the force on a particle at A , in the like direction, in the very same ratio of ED to AO , that is, in the ratio of the cosine of the arch AE to the radius. But this, by *Corol. I. Lem. II.* appears to be the law of the forces under which a ring of particles $AELaI$, detached from the earth, may continue in *equilibrio*, in the same plane, under a twofold motion about the center O , and about the line Aa as an axis.

Imagine now this ring to be exceeding dense, so that its *momentum of rotation* about its center O , may be equal to that of the earth itself, or so that the two bodies may equally endeavour to persevere in the same state and direction of motion, in opposition to any new force impressed. Then it is evident, that, were the forces whereby the two bodies tend to turn about the line LI , through the sun's attraction, to be *also* equal, the *same* effect, or alteration of motion, would be produced in both; and consequently, that the effects produced, when the forces applied are unequal, will be in proportion directly as the forces. Now the force whereby a particle at A is urged from the plane GG , is found to be $\beta \times \frac{3tt}{TT} \times \frac{AK}{AO}$ (by *Prop. I. Corol. I.*); which, in a direction perpendicular to the plane of the equator or ring, will be $\beta \times \frac{3tt}{TT} \times \frac{AK}{AO} \times \frac{OK}{AO} = \beta \times \frac{3tt}{TT} \times mn$.

Therefore, the force acting on a particle at E , in a like direction, being expressed by $\beta \times \frac{3tt}{TT} \times mn \times \frac{ED}{AO}$, the effect thereof to turn the ring about the line IL will be expressed by

by $\beta \times \frac{3^{tt}}{T^T} \times mn \times \frac{ED^2}{AO}$; which being as the square of the distance ED, it follows (*from the Corol. to Lem. IV.*) that, if M be taken to denote the mass of the ring, the whole force by which the ring tends to move about the line LI, as an axe, through the action of the sun on all the particles, will be truly defined by $\beta \times \frac{3^{tt}}{T^T} \times mna \times \frac{1}{2}M$. Again, because $\beta \times \frac{3^{tt}}{T^T} \times mn$ is the force acting on a particle at A, in a direction perpendicular to the plane of the ring, it is evident, *from Corol. I. to Lem. II.* that the ring will, in consequence of that force, have a motion about the line Aa as an axe; whose celerity will be to the celerity of the other motion about the center, in the proportion of r to 1, or of $\frac{3^{tt}}{2T^T} \times mn$ to 1; because, $\beta \times \frac{3^{tt}}{T^T} \times mn$ being $= \beta \times 2r$, r will here be $= \frac{3^{tt}}{2T^T} \times mn$. Therefore, if N be assumed to denote the sun's force to turn the earth about its center, we shall (*from the above observation*) have,
 $\beta \times \frac{3^{tt}}{T^T} \times mna \times \frac{1}{2}M$ (the force of the ring) : $N :: \frac{3^{tt}}{2T^T} \times mn$ (the motion of the ring) : $\frac{N}{\beta a M} =$ the required motion of the earth itself, about the same line Aa.

But it appears, *by the Corol. to Lem. V.* that the mass (M) of the ring (to have an equal momentum) must be just $\frac{2}{5}$ ths of the mass (S) of the earth: theref. our last expression is equal to $\frac{N}{\beta a \times \frac{2}{5}S}$; which,

by substituting $\beta \times \frac{3^{tt}}{T^T} \times \frac{mn \times aa - bb}{5a} \times S$, in the room of its equal

N , at last becomes $\frac{3^{tt}}{2T^T} \times \frac{mn \times aa - bb}{aa}$; being exactly the same as was before found in *Prop. II.* The ascertaining of which is the only real difficulty in the subject; since, that being once known, every thing that follows after is *purely* mathematical; nothing more being required than to take the sum, or fluent of those instantaneous alterations, in order to have the whole alteration, for any finite time proposed; as is actually done in *Prop. IV.* which therefore it will be needless to repeat.

SCHOLIUM.

Fig. 12. From this last method it will not be very difficult to determine what the result ought to be, when the density, instead of being every-where the same, is supposed to increase or decrease from the surface to the center, according to a given law. For, let N , as above, be taken to denote the force whereby the earth tends to turn about its center, by the action of the sun (the determination of which will be given by-and-by); and let M , also as before, express the quantity of matter in an exceeding dense ring, at the equator, having the same time of revolution, and *momentum of rotation*, with the earth itself; then it will appear, from the last Problem, that, let the figure and density of the earth be what they will, the celerity of the angular motion about the aforesaid line Aa , will be to that about the axis, in the ratio of $\frac{N}{\rho a M}$ to 1. Now, if the momentum of the earth about its axis (which I shall denote by R) be computed (by taking the sum of the products of all the particles by the squares of their, respective, distances from the axis) the value of M will be known; because the momentum of the ring (by the same rule) being $= M \times a^2$, we have $M \times a^2 = R$; and consequently $\frac{N}{\rho a M} = \frac{a N}{\rho R}$; so that, *the general proportion between the celerities of the two motions, about the line Aa , and the axe Pp , will be that of $\frac{a N}{\rho R}$ to unity.*

But now, in order to find these values of N and R , according to any hypothesis of density, we must look back to the third and fifth *Lemmas*; from the latter of which it appears, that the value of the momentum (R), when the density is uniform, will be truly defined by $\frac{8 \rho a^4 b}{15}$; a being the semi-equatoreal diameter, and b the semi-axis of the spheroid, and ρ the measure of the periphery of the circle whose radius is unity.

Now let us suppose the spheroid, instead of being every-where of the same density, to be composed of *elliptical strata*, whose densities vary according to any given law of the distances of such *strata* from the center of the spheroid.

Then,

Then, putting z = the semi-equatoreal diameter of any such stratum, and supposing the corresponding semi-axis to be in proportion thereto, as w to 1 (which proportion may be assumed, either, as constant or variable), we shall, by writing z in the room of a , and wz in the room of b , have $\frac{8p}{15} \times wz^5$; whose fluxion, or indefinitely small increment, will, therefore be the momentum of the stratum or shell, whereof the semi-equatoreal diameter is z , and thickness (at that diameter) $\dot{z}$; on the former supposition, that the spheroid is every-where of the same density. But in the present case this momentum must be drawn into the quantity, which, according to hypothesis, expresseth the measure of the density answering to the value of z ; which quantity we will represent by D ; then will the product $\frac{8pD}{15} \times \text{flux. } wz^5$ be the general fluxion of the momentum, when the density is variable: and therefore the fluent of this last expression, when $z = a$, and $w = \frac{b}{a}$, will be the true value of the required momentum (R) in the present case.

After the same manner, the corresponding value of N (from Lem. III.) may be determined: for, retaining the above notation and suppositions, it is evident (from thence) that the said value of N (which is there expressed by $\frac{\gamma}{a} \times \frac{mn}{5} \times \overline{a^4 - b^4} \times S$, or $\frac{\gamma}{a} \times \frac{mn}{5} \times \overline{a^4 - b^4} \times \frac{4pa^2b}{3}$) will here, by substitution, become $\frac{\gamma}{a} \times \frac{4mnp}{15} \times \overline{wz^5 - w^3z^5}$; and consequently that $\frac{\gamma}{a} \times \frac{4mnp}{15} \times D \times \text{flux. } \overline{wz^5 - w^3z^5}$ will be the required fluxion of the value of N: where $\frac{\gamma}{a}$ is a constant quantity (by hypothesis), the force being proportional to the distance, and the measure thereof (γ) at the given distance a , equal to $\beta \times \frac{3tt}{TT}$, as has been before shewn, in Prop. II.

Now from the whole of what is above laid down, it will appear evident, that the angular celerity of the motion about the line Aa (supposing that about the axis to be denoted by unity)

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unity) will be truly defined by $\frac{3mnt}{2T^2} \times \frac{\text{fluent of } D \times \text{flux. } wz^5 - w^3z^5}{\text{fluent of } D \times \text{flux. } wz^5}$; which, therefore, will be known, when the relation of z , w , and D is assigned.

If w be supposed constant, or, which is the same, if all the *strata* are conceived to be similar to one another, then our expression will become $\frac{3mnt}{2T^2} \times \frac{w - w^3 \times \text{fluent of } D \times \text{flux. } z^5}{w \times \text{fluent of } D \times \text{flux. } z^5} = \frac{3mnt}{2T^2} \times \frac{w - w^3}{w} = \frac{3mnt}{2T^2} \times \frac{1 - w^2}{1} = \frac{3tt}{2T^2} \times \frac{mn \times aa - bb}{aa}$ (because $w = \frac{b}{a}$): which conclusion appears to be the very same with that found when the density was supposed uniform. From whence it is evident that an increase or decrease of density, in going towards the center, makes no sort of difference here; provided the surfaces of the several *strata* are all similar to one another and to the surface of the earth. If indeed the *strata* are *dissimilar*, the case will be otherwise; as will be seen by the following example: which ought not be looked upon as a matter of mere speculation; since it will appear, in the sequel, that the precession of the equinox cannot be accounted for, so as to agree with the *phenomenon*, upon the supposition of an uniform density of all the parts of the earth; the result, this way, coming out about $\frac{1}{3}$ part greater than the real quantity, determined by observation.

Let then, as before, the greatest semi-diameter of any stratum be denoted by z , and let the least semi-diameter (lying in the axe of the earth) be in proportion thereto as $1 - \lambda z^q$ to unity; also let the density be supposed to increase, in approaching the center of the earth, in the ratio of $\pi - \frac{\pi - 1}{1} \times \frac{z^v}{a^v}$; so as to vary according to some power (z^v) of the distance, and that the measure thereof at the center, may be to that at the surface, in any given ratio of π to 1. Then, by taking λ , ϕ , and v , as constant quantities, and writing $1 - \lambda z^q$, and $\pi - \frac{\pi - 1}{1} \times \frac{z^v}{a^v}$, instead of their equals w and D , we shall here have

D

$$D \times \text{flux. } wz^5 - w^3z^5 = \pi - \pi - 1 \times \frac{z^v}{a^v} \times \phi + 5 \times 2\lambda z^{\phi+4}z^{\dot{z}},$$

nearly (because, to render the calculus less laborious, the terms involving λ^2 and λ^3 may be here neglected as inconsiderable); the fluent of which expression, when $z = a$, will be found

$$= \pi \times 2\lambda a^{\phi+5} - \frac{\pi - 1 \times \phi + 5 \times 2\lambda a^{\phi+5}}{\phi + v + 5} = \frac{v\pi + \phi + 5}{v + \phi + 5} \times 2\lambda a^{\phi+5}.$$

Moreover we have, in this case, $D \times \text{fluxion of } wz^5 =$

$$\pi - \pi - 1 \times \frac{z^v}{a^v} \times 5z^4\dot{z} - 5 + \phi \times \lambda z^{\phi+4}\dot{z} = \pi - \pi - 1 \times \frac{z^v}{a^v}$$

$\times 5z^4\dot{z}$, nearly (because $5 + \phi \times \lambda z^{\phi+4}$, as the earth is nearly spherical, is inconsiderable in respect of $5z^4\dot{z}$); whereof the

fluent, when $z = a$, will be had $= \pi a^5 - \frac{\pi - 1 \times 5a^5}{v + 5} = \frac{v\pi + 5}{v + 5} \times a^5$.

Now let these two values be substituted in the general expression

$$\text{on } \frac{3mntt}{2TT} \times \frac{\text{fluent of } D \times \text{flux. } wz^5 - w^3z^5}{\text{fluent of } D \times \text{flux. } wz^5}; \text{ by which means it be-}$$

$$\text{comes } \frac{3mntt}{2TT} \times \frac{v\pi + \phi + 5 \times v + 5}{v + \phi + 5 \times v\pi + 5} \times 2\lambda a^{\phi}. \text{ But, when } z = a,$$

$$w (= \frac{b}{a}) \text{ will be } = 1 - \lambda a^{\phi}, \text{ and } w^2 = 1 - 2\lambda a^{\phi}, \text{ nearly;}$$

$$\text{whence we have } 2\lambda a^{\phi} = 1 - w^2 = 1 - \frac{bb}{aa} = \frac{aa - bb}{aa}; \text{ and}$$

so, by substitution, our last formula becomes

$$\frac{3mntt}{2TT} \times \frac{v\pi + \phi + 5 \times v + 5}{v + \phi + 5 \times v\pi + 5} \times \frac{aa - bb}{aa}. \text{ Whence it appears that the}$$

required motion, in this case, is to the motion, when the den-

sity is supposed uniform, in the proportion of $\frac{v\pi + \phi + 5 \times v + 5}{v + \phi + 5 \times v\pi + 5}$

to unity.—From this proportion a great number of Corollaries may be drawn; but these will be, more properly, considered hereafter.

PROBLEM VI.

To find the quantity of the precession of the equinox, and also that of the nutation of the earth's axis, caused by the moon, during the time of half a revolution in her orbit.

Fig. 13. Let $fFN\epsilon E$ be the orbit of the moon (on the surface of the sphere) intersecting the ecliptic $\simeq \mathcal{V}\mathcal{S}\mathcal{Q}$ in N ; and suppose $F\epsilon DE\mathcal{Q}$ to be the position of the equator, on the moon's passing it at F , and $fbDea$ the position thereof when she repasseth it again at ϵ : let moreover the quantity of the annual precession arising from the sun (given by Prob. IV.) be denoted by A ; and let the ratio of the densities of the moon and sun be expressed by that of m to unity: then, taking τ to represent the given time in which the moon is moving from F to ϵ , the mean quantity of the precession, arising from the sun, in that time, will be $\frac{\tau}{T} \times A$: and therefore, since the perturbing forces of the sun and moon are as the densities (by Prob. I. Corol. II.) it is evident that the precession ($E\epsilon$) caused by the moon, in the same time, with respect to the plane of her own orbit, would be truly expressed by $m \times \frac{\tau}{T} \times A$, were the orbit's inclination to the equator to be always the same as that of the ecliptic to the equator: but, since the magnitude, as well as the position, of the angle E varies, with the place of the node, the said quantity $m \times \frac{\tau}{T} \times A$ must therefore be diminished in the ratio of the co-sine of E to the co-sine of $\mathcal{Q}$ (as appears by the said Prob. IV.) and then we shall get $\frac{m\tau A}{T} \times \frac{\text{co-sin. } E}{\text{co-sin. } \mathcal{Q}}$, for the true value of the precession $E\epsilon$, caused by the moon, with respect to her own orbit.

But now, in order to refer this to the ecliptic, it will be requisite to observe first of all, that, as the inclination of the earth's axis, at the end of every half revolution, on the return of the sun or moon again into the plane of the equator, is restored to its former quantity (by Corol. III. to Prob. IV.) it follows, seeing the angles E , ϵ , F , f are thus equal, that the triangles

triangles DEe and DfF will also be equal and alike, in all respects; and so, DE + De being = DE + DF = a semi-circle, both DE and De may be taken as quadrantal, or arcs of 90° each: whence, if $\angle R$, the measure of the angle φ , be supposed to meet aED in r, it will be, as *fin.* ED (*radius*)

$$: \text{fin. } e(E) :: Ee \left(\frac{m\tau A}{T} \times \frac{\text{co-fin. } E}{\text{co-fin. } \varphi} \right) : EDe = \frac{m\tau A}{T} \times \frac{\text{fin. } E \times \text{co-f. } E}{\text{co-f. } \varphi \times \text{rad.}};$$

also, as *fin.* a (φ) : *fin.* φD (co-fin. φE) :: EDe : $\varphi a = \frac{m\tau A}{T} \times \frac{\text{fin. } E \times \text{co-fin. } E \times \text{co-fin. } \varphi E}{\text{fin. } \varphi \times \text{co-fin. } \varphi \times \text{rad.}}$, the required quantity of the precession:

And, as *fin.* r (*radius*) : *fin.* DR (φE) :: EDe (RDr) : Rr = $\frac{m\tau A}{T} \times \frac{\text{fin. } E \times \text{co-fin. } E \times \text{fin. } \varphi E}{\text{co-fin. } \varphi \times \text{rad.}}$, the corresponding quantity of the nutation, or the decrease of the inclination of the equator to the ecliptic. Q. E. I.

COROLLARY.

It is evident from hence that the quantity of the nutation is to that of the precession, as $\frac{\text{fin. } \varphi E}{\text{rad.}}$ to $\frac{\text{co-fin. } \varphi E}{\text{fin. } \varphi}$, or as *fin.* φ to $\frac{\text{rad.} \times \text{co-fin. } \varphi E}{\text{fin. } \varphi E}$; that is, as the sine of φ to the co-tangent of φE . It appears moreover (because *fin.* E : *fin.* φN :: *fin.* N : *fin.* φE , *p. spherics*) that the former of these quantities is also truly explicable by $\frac{m\tau A}{T} \times \frac{\text{fin. } N \times \text{fin. } \varphi N \times \text{co-fin. } E}{\text{co-fin. } \varphi \times \text{rad.}}$: which expression will be of use in the following Problem.

PROBLEM VII.

To determine the precession of the equinox, and the quantity of the nutation of the earth's axis, caused by the moon, during the time of half a revolution of the node of the moon's orbit.

Things being supposed as in the preceding Problem, let the distance φN of the node from the equinoctial point be denoted by z , its sine by x , and its co-sine by y ; let also the sine of the angle $N\varphi E$ be put = a , its cosine = b , the sine of

N

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$N = c$, its co-sine $= d$, the semi-periphery $\Upsilon \mathcal{R} \Xi = e$, and the time of half a revolution of the node $= R$.

If φQ be supposed perpendicular to NE , it will be, *p. spherics*, as co-fin. $N\varphi (y) : \text{radius} (1) :: \text{co-tang. } N \left(\frac{d}{c} \right) : \text{tang. } N\varphi Q = \frac{d}{cy}$; let this be denoted by b , then the secant of the same angle will be, $= \sqrt{1+bb}$, its sine $= \frac{b}{\sqrt{1+bb}}$, and its co-sine $= \frac{1}{\sqrt{1+bb}}$: whence, by the known rules for finding the sine and co-sine of the difference of two angles, the sine of $E\varphi Q$ will also be had $= \frac{hb-a}{\sqrt{1+bb}}$, and its cosine $= \frac{ha+bb}{\sqrt{1+bb}}$. Whence, again, *p. spherics*, we have, as fin. $N\varphi Q : \text{fin. } E\varphi Q :: \text{co-fin. } N : \text{co-fin. } E = \frac{bb-a \times d}{b} = bd - \frac{ad}{b} = bd - acy$; also, as co-fin. $N\varphi Q : \text{co-fin. } E\varphi Q :: \text{co-tang. } N\varphi \left(\frac{y}{x} \right) : \text{cotang. } \varphi E = \overline{b+ta} \times \frac{y}{x} = \frac{ad+bcy}{cx}$; because $b = \frac{d}{cy}$.

But, by the Corol. to the last Problem, the quantity of the nutation for the time τ , is expressed by $\frac{m\tau A}{T} \times \frac{\text{fin. } N \times \text{fin. } \varphi N \times \text{co-f. } E}{\text{co-fin. } N \varphi E \times \text{rad.}^2}$; which, in algebraic terms (by substituting the above values), will become $\frac{m\tau A}{T} \times \frac{cx \times \overline{bd - acy}}{b}$. But the time τ , during which the longitude (z) of the node is increased by $\dot{z}$, being to R the time of half a revolution of the node, as $\dot{z}$ to e , its value will therefore be expounded by $R \times \frac{\dot{z}}{e}$, or its equal $R \times \frac{\dot{z}}{e\sqrt{1-xx}}$: and so by substituting this value, and writing $\sqrt{1-xx}$ in the room of y , our last expression will be reduced to $\frac{mAR}{T} \times \frac{c}{eb} \times \frac{bdxx}{\sqrt{1-xx}} - acxx$; whose fluent $\frac{mAR}{T} \times \frac{c}{eb} \times bd - bd\sqrt{1-xx} - \frac{1}{2}acxx$ ($=$

($= \frac{mAR}{T} \times \frac{c}{be} \times \overline{bd \times \text{versed sine } z - \frac{1}{4}ac \times \text{versed sine } 2z}$) is consequently the measure of the nutation, or the decrease of the inclination of the equator to the ecliptic, caused by the moon, from the time of the node's coinciding with the equinoctial point φ , to its arrival at the position N .

Again, with regard to the precession of the equinox, the increase thereof being in proportion to the decrement of the inclination, as the co-tangent of φE to the sine of $N\varphi E$ (by the Corollary to the precedent) or, in species, as $\frac{ad + bcy}{cx}$ (or $\frac{ad + bc\sqrt{1-xx}}{cx}$) to a , its fluxion will therefore be had by mul-

tiplying that of the nutation, given above, into $\frac{ad + bc\sqrt{1-xx}}{acx}$, and so is found to be

$$\frac{mAR}{T} \times \frac{1}{abe} \times \frac{abd^2x}{\sqrt{1-xx}} + \overline{bb - aa \times cdx - abc^2x\sqrt{1-xx}};$$

whose fluent, which is

$$\frac{mAR}{T} \times \frac{1}{abe} \times \overline{abd^2z + bb - aa \times cdx - \frac{1}{2}abc^2z - \frac{1}{2}abc^2x\sqrt{1-xx}}$$

$$(\overline{= \frac{mAR}{T} \times \frac{1}{abe} \times dd - \frac{1}{2}cc \times abz + bb - aa \times cd \times f.z - \frac{1}{4}abcc \times f.2z}),$$

must consequently be the precession itself.

But, at the end of half a revolution, when the node N arrives at the other equinoctial point ω , both this, and the expression for the nutation, will become much more simple, x being then $= 0$, and $z = e$; whence the nutation will be $=$

$$\frac{mAR}{T} \times \frac{c}{eb} \times 2bd = \frac{mAR}{T} \times \frac{2cd}{e}; \text{ and the precession equal to}$$

$$\frac{mAR}{T} \times \overline{dd - \frac{1}{2}cc} = \frac{mAR}{T} \times \overline{1 - \frac{1}{2}cc} \text{ (because } cc + dd = 1).$$

Q. E. I.

COROLLARY I.

It appears from hence, that the mean precession of the equinox, arising from the action of the moon, is in proportion to what it would otherwise be, if the moon's orbit was to coincide with the ecliptic, as $1 - \frac{1}{2}cc$ to unity: whence the true value:

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value thereof is to that depending on the sun, in a ratio compounded of the ratio of the density of the moon to the density of the sun, and the aforesaid ratio of $1 - \frac{1}{2}cc$ (or 0,988) to unity.

COROLLARY II.

It appears likewise, that the whole quantity of the nutation, in half a revolution of the node, is to the corresponding quantity of the precession, as $\frac{2cd}{e}$ to $dd - \frac{1}{2}cc$, or as unity to

$\frac{d}{c} - \frac{\frac{1}{2}c}{d} \times \frac{e}{2}$; that is, as the radius to the excess of the cotangent, above half the tangent of the orbit's inclination, drawn into (1,5708) the measure of half the periphery of the circle whose diameter is unity. This proportion, in numbers, supposing the mean inclination of the orbit to be $5^\circ 8'$, will be found to be as 10 to 174, very near.

COROLLARY III.

Moreover, seeing the precession in half a revolution of the node is $\frac{mAR}{T} \times dd - \frac{1}{2}cc$, we have, as $e : z :: \frac{mAR}{T} \times dd - \frac{1}{2}cc$: $\frac{mAR}{T} \times \frac{z}{e} \times dd - \frac{1}{2}cc$, the quantity of the mean precession during the time in which the node moves over the arch z , or φN . This being subtracted from the true precession, found above, the remainder

$\frac{mAR}{T} \times \frac{1}{abe} \times \overline{bb - aa \times cd} - \frac{1}{2}abc^2 \times \sqrt{1 - xx}$ will consequently be the equation of the precession, or the excess of the true above the mean: which equation or excess, if we neglect the term $-\frac{1}{2}abc^2 \times \sqrt{1 - xx}$ (whose value, by reason of the smallness of c^2 , never amounts to $\frac{1}{4}$ th of a second) will evidently be at its greatest value at the end of $\frac{1}{4}$ th of a revolution, on the node's arrival at the solstice; when it becomes $\frac{mAR}{T} \times \frac{1}{abe} \times \overline{bb - aa \times cd}$; and, is therefore, in proportion to $\frac{mAR}{T} \times \frac{2cd}{e}$, the whole, or greatest quantity of the nutation, during half a revolution of the node, as $bb - aa : 2ab$, or

as

as $1 : \frac{2ab}{bb - aa}$, that is, as the radius to the tangent of double the inclination of the equator to the ecliptic.

COROLLARY IV.

Furthermore, since the value of c (the sine of the orbit's inclination) is but small, the last term of the general expression for the nutation, as well as that for the excess of the true precession above the mean, may be rejected, without producing any considerable error; whence the nutation is reduced to $\frac{mAR}{T} \times \frac{cd}{e} \times 1 - \sqrt{1 - xx}$, and the precession to $\frac{mAR}{T} \times \frac{1}{abe} \times \overline{bb - aa} \times cdx$. Hence it appears that the decrease of the inclination, from the time of the node's leaving the equinoctial point γ , will be as the versed sine $(1 - \sqrt{1 - xx})$ of the node's true longitude; and that the excess of the true precession above the mean, will be always as the sine (x) of the same longitude.

SCHOLIUM.

The quantity of the annual precession of the equinox arising from the force of the sun, is found in Prob. IV, to be $21'' 7'''$; upon the supposition of all the parts of the earth being homogenous, and in a state of fluidity. If, therefore, this quantity be taken from $(50'')$ the whole, *observed*, annual precession, arising from the sun and moon conjunctly, the remainder $28'' 53'''$ will consequently be the mean annual precession depending on the moon; which being increased in the ratio of 1000 to 988 (according to Prob. VII. Corol. I.) gives $29'' 14'''$, for the quantity of the precession, if the orbit of the moon were to coincide with the plane of the ecliptic. Hence it will be (by the same Corol.) as $21'' 7'''$ is to $29'' 14'''$, so is the density of the sun to the density of the moon, *according to this hypothesis*. But it is evident from experience (whether we regard the proportion of the tides, or the accurate observations of Dr. BRADLEY) that the density of the moon in respect to *that* of the sun, cannot be so small as it is here assigned.

It is true, there is no way of knowing the *exact* ratio of the densities of the two luminaries; since *theory*, for want of sufficient *data*, fails us here. And as to the method, by *observing and comparing the spring and neap tides* * (whether we regard the quantities or times of *them*) it cannot be otherwise than very precarious; considering the many obstacles and intervening causes by which *they* are perpetually, more or less, influenced and disturbed. Upon the whole it therefore seems to me, that the best method to settle this point (as far as the nature of the subject will allow of) is from the *observed* quantity of the nutation itself; agreeable to what has been hinted on this head by that celebrated Astronomer, to whose accurate observations we owe this important discovery.

Let us, therefore, take g to denote the greatest nutation of the earth's axis, *as given by observation*; and then, if f be taken to represent the mean annual precession, *given in like manner*, it will appear (by *Prob. VII. Corol. II.*) that $\frac{174}{10} \times \frac{1}{9\frac{1}{3}} \times g$ is the part of the said annual precession depending on the moon; whence the remaining part, owing to the sun, must necessarily be $f - \frac{174}{93} \times g (= \frac{31f - 58g}{31})$. Therefore we have (by *Problem VII. Corol. I.*) as $\frac{58g}{31} \times \frac{1000}{988}$ to $\frac{31f - 58g}{31}$, or as $\frac{58g}{31f - 58g} \times \frac{1000}{988}$ to 1, so is the density of the moon to the density of the sun; which, in numbers (making $g = 18''$), will come out as 2,09 to 1. But if the value of g be supposed only a second or two greater or less than $18''$, the result will be sensibly different, as may be seen in the annexed Table; wherein, besides the ratio of the densities, are also exhibited the mean quantities of the annual precession, depending on the forces of the sun and moon, respectively; together with the greatest equation of the said precession, *as given by Problem VII. Corollary III.*

* Sir ISAAC NEWTON, by this method, makes the proportion to be as $4\frac{1}{2}$ to 1; and M. DANIEL BERNOULLI, only as $2\frac{1}{2}$ to 1.

Greatest nutation.	Ratio of the densities of the sun and moon.	Annual precession caused by the sun.		Mean annual precession caused by the moon.		Greatest equation of the precession caused by the moon.	
		Seconds	Thirds	Seconds	Thirds	Seconds	Thirds
16	1 : 1,51	20	3	29	57	14	58
17	1 : 1,77	18	11	31	49	15	54
18	1 : 2,09	16	19	33	41	16	50
19	1 : 2,50	14	27	35	33	17	46
20	1 : 3,01	12	35	37	25	18	42

Were I to deliver my opinion which of the different cases here put down answers best to the *phenomenon*, and the general law of gravitation, I should, without hesitating, fix upon that preceding the last; which, upon the whole, will be found to agree better with Dr. BRADLEY's observations than any of the others: besides, though the observations on the tides cannot be relied on to any great degree of exactness, yet, by them, it is sufficiently evident, that the perturbing force of the moon cannot be to *that* of the sun in a less proportion than of about $2\frac{1}{2}$ to 1.

From the greatest nutation, and the greatest equation of the precession, *given above*, the quantity of the nutation and the equation of the precession, corresponding to any given position of the lunar node, may be very easily determined: for, *first*, it will be, *by Corol. IV.*

As the radius is to the sine of the node's distance from the nearest equinoctial point, so is the greatest equation of the precession to the equation sought.

Which must be added to the mean precession when the node (*viz.* the ascending one) is in any of the six southern signs; but subtracted, when in any of the six northern ones.

Secondly, it will appear, *by the same Corollary*, that the decrease of the inclination of the equator to the ecliptic, from the time of the node's coinciding with the equinoctial point ♈, is proportional to the versed sine of the node's present distance from that point: whence it follows, that the said inclination will be at its mean value when the node is in the solstice; and consequently, that the difference between the mean, and true

values, will be as the difference between the verfed sine of the node's present distance from φ , and the verfed sine of 90 degrees, that is, as the co-sine of the node's distance from φ . Therefore, to find the nutation at any given time, it will be,

As the radius is to the co-sine of the node's distance from the nearest equinoctial point, so is the greatest nutation to the nutation sought.

Which, to have the true obliquity of the equator to the ecliptic, must be added, when the node is in any of the six ascending signs $\mathbb{W}$, $\mathbb{W}$, $\mathbb{X}$, φ , δ , Π ; but, otherwise, subtracted.

The following Table, shewing by inspection, as well the equation of the precession, as that of the obliquity of the ecliptic, is computed from the proportions here laid down; upon supposition that the greatest quantity of the nutation is 19 seconds.

The Equation of the Precession of the Equinox.					The Equation of the Obliquity of the Ecliptic.				
φ 's δ	Sig. 0	Sig. I	Sig. II	Subtr.	φ 's δ	Sig. 0	Sig. I	Sig. II	Add.
from φ	Sig. VI	Sig. VII	Sig. VIII	Add.	from φ	Sig. VI	Sig. VII	Sig. VIII	Subtr.
Deg.	Seconds	Seconds	Seconds	Deg.	Deg.	Seconds	Seconds	Seconds	Deg.
0	0,0	8,8	15,3	30	0	9,5	8,2	4,7	30
5	1,5	10,1	16,1	25	5	9,4	7,8	4,0	25
10	3,0	11,4	16,7	20	10	9,3	7,3	3,3	20
15	4,5	12,5	17,2	15	15	9,2	6,7	2,4	15
20	6,0	13,6	17,5	10	20	9,0	6,1	1,7	10
25	7,4	14,5	17,7	5	25	8,7	5,5	0,9	5
30	8,8	15,3	17,7	0	30	8,2	4,7	0,0	0
Subtr.	Sig. V	Sig. IV	Sig. III	φ 's δ	Subtr.	Sig. V	Sig. IV	Sig. III	φ 's δ
Add.	Sig. XI	Sig. X	Sig. IX	from φ	Add.	Sig. XI	Sig. X	Sig. IX	from φ

Fig. 15. To place what has been delivered above in another view, suppose PE to be equal to the mean distance of the pole of the equator from the pole E of the ecliptic; in which (produced) let there be taken PA and PB equal, each, to half the greatest nutation; and about AB, as an axis, conceive an ellipse ACBF to be described; whose other axe CF is to AB in the ratio of the co-sine of $2EP$ to the co-sine of EP (that is, in numbers, as 7444 to 10000, or as 3 to 4, nearly); then, if the point P represents the mean place of the pole of the equator, the true place will always be found in the circumference of the said ellipse. And if, on the diameter AB, a circle ADBG be also described, and the angle APS be made equal to the distance of the node from the equinoctial point φ ; then, I say, a perpendicular $SR\varphi$, falling from the point S upon the diameter AB, will

will intersect the circumference of the ellipse in the point where the pole of the equator (p) is, at that time, posited. For, first, it is clear, from what has been already remarked, that AE and BE will be the greatest and least distances of the two poles, as being equivalent to the respective inclinations of the two planes, the equator and the ecliptic: from whence and Corol. IV. it is manifest, that ER or Ep will be the true distance of the said poles, when the versed sine of the node's distance (APS) from φ is AR. Moreover, by construction, $CP : AB :: \frac{1}{2}$ the co-sine $2EP : \text{co-sin. EP}$; that is, in species, $CP : AB :: \frac{bb - aa}{2} : b$. And, p . *Spherics*, tang. PEC : tang. PC ($:: PEC : PC$, nearly) $:: \text{rad. (1)} : a$. Therefore, by compounding these two proportions, we have $PEC : AB :: \frac{bb - aa}{2} : ab :: bb - aa : 2ab$: which proportion, for finding the angle PEC, is the very same with that determining the greatest difference of the mean, and true longitudes, as given by Corol. III. Whence it easily follows, that the angle REp will express the difference of the mean and true longitudes, at the given position of the node; since, as the radius : sine APS ($:: PD : RS :: PF : Rp$) $::$ the angle PEC : the angle REp, as it ought to be, by Corol. IV. The ratio of CF to AB is here determined to a geometrical exactness, as no-ways depending, either, on the density of the moon, or on any other physical hypothesis.

Having now laid down the general proportions for the nutation of the earth's axis, and the precession of the equinox, I shall here subjoin the necessary rules for determining how much the declinations and right-ascensions of the stars are affected by those inequalities.

1°. For the alteration of a star's declination, and right-ascension, arising from the nutation of the earth's axis; it will be

As the radius is to the sine of the star's right-ascension, so is the nutation (or the given alteration of the equator's inclination to the ecliptic) to the alteration of the star's declination, caused by the nutation;

And, as the co-tangent of the star's declination is to the co-sine of its right-ascension, so is the nutation to the alteration of the star's right-ascension, corresponding.

2°. For

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2°. For the alteration of the star's declination and right-ascension, arising from the precession of the equinox; it will be

As the co-secant of the obliquity of the ecliptic is to the co-sine of the star's right-ascension, so is the precession of the equinox (or the alteration of the star's longitude) to the alteration of the star's declination, caused by the precession;

And as the co-sine of the star's declination is to the co-tangent of its angle of position, so is the alteration of declination, found by the last proportion, to the alteration of right-ascension, answering thereto.

Any one, but little acquainted with the sphere, will easily see *when* these equations are additive, and *when* subtractive: nor will it be at all difficult to comprehend the reasons upon which they are founded; they being nothing more than so many particular cases of the general relation subsisting between the fluxions of the sides and angles of a spherical triangle *. It will not, however, be improper to remark here, that, when the quantity of the precession, in the second of the preceding cases, amounts to some *minutes*, it will be necessary, in order to have the conclusion sufficiently exact, to make use of the mean right-ascension, at the middle of the given interval; which, from the given right-ascension at the beginning of the interval, may be estimated near enough for the purpose, in most cases, without the trouble of a calculation: but in other cases, and when the utmost exactness is required, it will be necessary to repeat the operation.

It may not be improper to observe likewise, that, besides the equations depending on the position of the lunar nodes, computed above, there is a small motion of nutation and precession arising from the moon's declination; whereof the greatest quantity is to the greatest quantity of *that* depending on the sun, in a ratio compounded of the ratio of the densities of the two bodies, that of their periodic times, and that of the sines of the inclinations of their respective orbits to the plane of the equator, nearly (as appears by Prob. IV. and VI.) Whence it is evident, that this part of the nutation, depending on the moon's declination, cannot, in any circumstance, amount to more than about $\frac{1}{4}$ th of a *second*; a quantity too small to merit attention in the practice of Astronomy.

* See my *Doctrine of Fluxions*, Part II. Sect. I.

Remarks on some Particulars in the preceding Theory and Calculations; in order to explain and obviate certain difficulties and objections that may thence arise.

IT may be observed, in the first place, that we have, all along, considered the effects of the sun and moon separately; and, consequently, have supposed them to be no-ways influenced or disturbed by each other. This may seem too bold an assumption; especially, as it is known that the tides, which are produced by the very same forces, depend upon, and are greatly varied by, the different positions of the two luminaries.

To remove this objection, let $\varphi SM \simeq$ represent the plane Fig. 16. of the earth's equator $\varphi O \simeq$, its intersection with the plane of the ecliptic, φS the right-ascension of the sun, and φM the right-ascension of the moon; and let the forces of the two bodies to turn the earth about its center, in those positions, be represented by f and F , respectively.

These forces may be considered as acting perpendicular to the plane of the equator in the points S and M , and will be equivalent to, and have the same effect with, one single force, equal to them both, acting in their center of gravity N . But, by mechanics, the force $f + F$, acting at N , will (if the radius OP be drawn through N) be equivalent to another force, acting at P , expressed by $\overline{f + F} \times \frac{ON}{OP}$, or $\overline{f + F} \times \frac{NQ}{PR}$ (supposing NQ , PR , as also SB and MC , to be perpendicular to $\varphi O \simeq$).

But the quantity of the precession, during a given moment of time, is known to be as the force, and as the sine of the right-ascension, conjunctly (*by Prob. III.*); from whence the two quantities arising from the sun and moon, considered separately, are expounded by $f \times SB$, and $F \times MC$, respectively. But, supposing both bodies to act together, or, which is the same, supposing one single force, expressed by $\overline{f + F} \times \frac{NQ}{PR}$,
to

to act at P, the quantity of the precession will then (by the very same rule) be truly defined by $\overline{f + F} \times \frac{NQ}{PR} \times PR$, or its equal $\overline{f + F} \times NQ$; which quantity, by the property of the center of gravity, is known to be equal to $f \times SB + F \times MC$. Hence it is manifest that, whether the forces of the luminaries be joined together, or treated apart, the result will be the same.

The next difficulty, relates to the excentricity of the lunar orbit, and the inequality of the motion in that orbit; which may be thought sufficient to occasion a sensible deviation from rules founded on a supposition that pays no regard to them.

Fig. 17. In order to clear up this point also, imagine ADBE to be an ellipse, in which the moon is supposed to revolve, about the center of the earth, placed in the lower focus F of the ellipse: let AB be the transverse axis of the ellipse, perpendicular to which, through F, draw the ordinate IH; moreover let there be drawn any two other lines DE, *de*, through the focus F, to make a very small (given) angle DF*d* with each other.

The perturbing force of the moon, at the distance DF, will (by *Prop. I. Corol. II.*) be, *inversely*, as the cube of that distance; and the time of describing the given angle DF*d* will, it is well known, be *directly* as the square of the same distance. Therefore, by composition, the quantity of the moon's action, during the time of describing this angle, will be in the simple ratio of the said distance, *inversely*. Hence it appears, that the sum of the forces employed, during the times of describing the opposite angles DF*d*, EFe, will be truly defined by $\frac{1}{FD} + \frac{1}{FE}$,

or its equal $\frac{FE + FD}{FE \times FD}$.

Upon AB let fall the perpendiculars DN and EM; so shall $FE - FH : FI (FH) - FD :: FM : FN$ (*p. ellipse*) :: $FE : FD$ (*p. sim. triang.*): consequently $FE \times FD - FH \times FD = FH \times FE - FD \times FE$, or $2FE \times FD = FH \times \overline{FE + FD}$: therefore, as it appears from hence that $\frac{FE + FD}{FE \times FD}$, the measure of the said forces, is, every-where, equal to the constant quantity

ity $\frac{2}{FH}$, it is evident that the excentricity of the orbit and the position of the apogee have no effect on the motion of the earth's axis.

An objection may, perhaps, arise, with regard to the addition of the forces employed by the moon in opposite parts of her orbit; which step may be looked upon as arbitrary: but the reason upon which it is founded will be clear, by considering that the moon's inclination to the plane of the equator, in opposite points of her orbit, is always the same; and that, therefore, the very same effect in the alteration of the position of the equator will be produced, whether the whole force employed during the description of the corresponding opposite angles, be equally, or unequally, divided, with respect to the said angles; since the said force acts with the same advantage, or under the same circumstance of declination, in both cases.

Another difficulty that may arise, is in relation to our having made the effect of the sun's force to be about $\frac{1}{3}$ part less than the quantity resulting from calculations founded on hydrostatical principles and the hypothesis of an uniform density of all the parts of the earth. But, that the *phenomenon* cannot be truly accounted for, upon this hypothesis, appears from the concurrence of all experiments in general: for, whether we regard the mensuration of the degrees of the earth, the accurate observations of Dr. BRADLEY, or the proportions and times of the tides, the case is the same, and requires a much less effect from the action of the sun than results from, or can consist with, the said hypothesis.

But if the density of the earth, instead of being uniform, is supposed to increase from the surface to the center (as there is the greatest reason to imagine it does), then the *phenomenon* may be easily made to quadrate with *the principles of gravitation*; and that according to innumerable suppositions, respecting the law whereby the density may be conceived to increase.

Thus, conformable to the hypothesis laid down in the Scholium after Prob. V. the motion of the equinoctial points will be in proportion to the motion of the same points, when the

density is supposed uniform, as $\frac{v\pi + \phi + 5 \times v + 5}{v + \phi + 5 \times v\pi + 5}$ to 1, that is,

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as $1 - \frac{v\phi \times \pi - 1}{v + \phi + 5 \times v\pi + 5}$ is to 1: therefore, by making $1 - \frac{v\phi \times \pi - 1}{v + \phi + 5 \times v\pi + 5} = 1 - \frac{1}{3}$ (agreeable to what has been above

observed), we shall have $\frac{v\phi \times \pi - 1}{v + \phi + 5 \times v\pi + 5} = \frac{1}{3}$: by means of which equation, the relation of π , ϕ , and v may be so assigned, as to give the true quantity of the precession, and that innumerable ways. As one instance hereof, let us suppose $\pi = 2$, or that the density at the center is just double to *that* at the surface; and let the value of ϕ be supposed very great, or, which comes to the same, let the *strata* in the lower parts of the earth, be supposed very nearly spherical, or orbicular: then our equation will become $\frac{v\phi}{v + \phi + 5 \times 2v + 5} = \frac{1}{3}$; which,

because ϕ is supposed very great, will be $\frac{v}{2v + 5} = \frac{1}{3}$, nearly; whence v is given $= 5$: so that, according to this hypothesis, the decrease of density, in going from the center of the earth to the surface, will be in the quintuplicate ratio of the distances from the center.

No One can imagine that we pretend here to ascertain the structure and density of the interior parts of the earth: all that is attempted, is to shew (which indeed is all that can be done) that the precession of the equinox may be truly accounted for *upon the principles of gravitation*, though not in the hypothesis of an uniform density of all the parts of the earth, unless by assuming the difference of the least and greatest diameters much smaller than it is found to be, either, from hydrostatical principles, or by an actual mensuration of the degrees of the earth's meridian.

There remains yet another particular that I cannot avoid taking some further notice of; which is the wide difference to be found between our conclusion, in Prob. IV. Corol. I. and *that* brought out by Sir ISAAC NEWTON (*in Prop. 39. Book III. of his Principia*) from the very same data.

I am

I am sensible that this is a delicate point to touch upon; but then I know likewise, that I might leave my Readers dissatisfied, were not I to endeavour to point out the causes of the said difference.—At first I had, myself, a strong suspicion that I had, somewhere, fallen into an error; which put me upon attempting the solution by different methods, as the most proper way to arrive at certainty, and to discover the mistake, if any *such* had crept into my calculations. Two of these methods I have given; the others seemed unnecessary. The exact concurrence of them all, first made me think, that it was not impossible but there might be a fault in that Author's solution; and occasioned my looking into his method with a more particular attention than I had before regarded it with.

What, at first, seemed most doubtful to me was his hypothesis, *that the motion of the nodes of a ring would be the same, whether the ring were fluid, or whether it consisted of a hard rigid matter* *: this, I say, did not seem at all clear, at first; but upon recollecting the demonstration of my second Lemma (wherein this point is fully, though not directly, proved) I was soon convinced that the fault (if such there was) must be owing to something else.

In the next place, his third Lemma did not appear to me so well grounded as the two preceding ones. In this Lemma

* The celebrated mathematician M. D'ALEMBERT, who has with great subtlety expatiated on Sir ISAAC NEWTON's solution of this Problem, represents the above hypothesis, as ill founded; and says, that, when the ring is in a fluid state, the particles, or detached moons will not have their centers in one and the same plane (*il est certain que des lunes isolées n'auront pas toujours leurs centres placés dans un même plan*). Now if, by this, we are to understand, that the deviation from a plane is something sensible in comparison of the nutation in question, what is advanced is repugnant to what is demonstrated in our second Lemma. But if an exceeding small deviation (depending on the second term of a series) be only intended (and such it must be, if any thing at all), such a supposition will make nothing against our Author's assumption; as, in physical subjects, a perfect accuracy is not to be expected. This learned gentleman himself allows, that, the considering of all the particles (or the ring of moons) as being in the same plane, produces no error in the conclusion: from whence it might, with some reason, be imagined, that the hypothesis itself could not be otherwise than true. And it seems farther plain to me, that, whatever lights that Author's oversights in the solution of this Problem are capable of being placed, his real mistakes are two only.

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he determines, *that the motion of the whole earth about its axis, arising from the motion of all the particles, will be to the motion of a ring about the same axis, in a proportion compounded of the proportion of the matter in the earth to the matter in the ring, and of the number 925725 to the number 1000000.* This proportion is, indisputably, true, in the sense of the Author: but there is a difference between the quantity of motion, so considered, and the *momentum* whereby a body, revolving round an axis, endeavours to persevere in its present state of motion, in opposition to any new force impressed. Now it seems clear to me, that it is this last kind of *momentum* that ought to be regarded, in computing the alteration of the body's motion, in consequence of any such force. And here every particle is to be considered as acting by a lever terminating in the axis of motion: so that, to have the whole momentum sought, the moving force of each particle must be multiplied into the length of the lever by which it supposed to act: whence the momentum of each particle will be proportional to the square of the distance from the axis of motion; as it is known to be in finding the centers of percussion of bodies, which depend on the very same principles.

Now, according to this way of proceeding, it will be found, that the momentum of the whole earth (taken as a sphere) will be to the momentum of a very slender ring, of the same diameter, revolving in the same time, about the same axis, in a proportion compounded of the proportion of the matter in the earth to the matter in the ring, and of the number 800000 to the number 1000000. Which proportion, therefore, differs from that of Sir ISAAC NEWTON, given above, in the ratio of 800000 to 925725: so that, if his result, which is $9^{\circ} 7' 20''$, be increased in this ratio, we shall then have $10^{\circ} 33''$, for the quantity of the annual precession of the equinox, arising from the force of the sun; allowing for the above-mentioned difference.

It appears further, by perusing his 39th Proposition, that he there assumes it as a principle, that, if a ring, encompassing the earth, at its equator *AlaL* (but detached therefrom) was to tend, or begin, to move about its diameter *LI* with the same accelerative

tive force, or angular celerity, as *that* whereby the earth itself tends to move about the same diameter, through the action of the sun (at S); that *then* the motion of the nodes of the ring and of the equator would be exactly *the same*. Now this Fig. 8. would indeed be the case, were not the effects of these forces whereby the two bodies tend to move about the diameter LI, to be influenced and interrupted by the other motions about the axe of rotation Pp, and that according to a different ratio, depending on the different figures of the earth and ring.

A sphere, let the direction of its rotation be which way it will, that is, let it move about what diameter it will, has always the *same momentum*, provided it has the same angular celerity: but the *momentum* of a very slender ring, revolving about one of its diameters, appears (*by Lem. IV.*) to be only the half of what it would be, if the revolution was to be performed in a plane, about the center of the ring. Whence it is evident, that the ring AlaL, to acquire the same motion of precession and nutation with the earth's equator, ought to tend to move about the diameter LI with an accelerative force double to *that* whereby the earth itself tends to move about the same diameter, through the action of the sun: since, in this case, the quantities of motion, or the *momenta*, generated in the two bodies, during any very small particle of time, would be exactly proportional to the respective *momenta* of rotation, whereby the bodies endeavour to persevere in their present state and direction of motion, in opposition to any new force impressed. Hence it follows that all conclusions, relating to the change of the position of the earth's axe, drawn from the principle above specified, must be too little by just one half; and consequently that the quantity of the annual precession of the equinox, arising from the action of the sun, ought to be the double of $10'' 33'''$; which is $21'' 6'''$, and agrees, to a *third*, with what we before found it to be, by two different methods.



A very exact METHOD for finding the PLACE of a PLANET in its orbit, from a Correction of WARD'S *hypothesis*, by means of One, or more Equations, applied to the motion about the upper focus.

Fig. 18.  ET ABPC be the ellipses in which the planet revolves about the sun in the lower focus S ; let F be the upper focus, and M and *m* any two places of the planet indefinitely near to each other ; and let FM, *Fm*, SM, *Sm* be drawn, as likewise MN, perpendicular to the greater axis AP : from the center F, with the radius FD = 1, let the circumference of a circle DEe be described, and from its intersection with FM draw EH perpendicular to AP : put AO (= OP) = *a*, OB (= OC) = *b*, OF (= OS) = *c*, FM = *u*, FH = *x*, EH = *y*, DE = *z*, and Ee = *ż* : then SM being (= AP — FM) = *2a — u*, by the nature of the ellipses, and FN (= FH × $\frac{FM}{FE}$) = *xu*, we have, by a known property of triangles, $\overline{SM + FM} \times \overline{SM - FM} (2a \times 2a - 2u) = 2SF \times ON (4c \times c + xu)$: from which equation *u* (FM) is found = $\frac{a^2 - c^2}{a + cx} = \frac{bb}{a + cx}$ (because *bb* = *aa — cc*) : whence SM (= *2a — u*) is also had in terms of *x*, being = $\frac{aa + cc + 2acx}{a + cx}$.

Now the area EFe being expressed by $\frac{1}{2}z\dot{z}$ (= FE × $\frac{1}{2}$ Ee), we have FE² (1) : FM² :: $\frac{1}{2}z\dot{z}$: the area MF*m* = $\frac{1}{2}z\dot{z} \times \overline{FM}^2$. Therefore, the angle SM*m* being equal to FMA, or F*m*A (by the nature of the curve) we also have FM × M*m* : Sm × *m*M (or FM : SM) :: the area ($\frac{1}{2}z\dot{z} \times \overline{FM}^2$) of the triangle FM*m* : the area of the triangle SM*m* = $\frac{1}{2}z\dot{z} \times FM \times SM$ (Elem. 23. 6.)

$$= \frac{\frac{1}{2}b^2z \times aa + cc + 2acx}{a + cx} = \frac{1}{2}b^2z\dot{z} + \frac{\frac{1}{2}b^2c^2z \times 1 - xx}{a + cx} = \frac{1}{2}b^2z\dot{z} + \frac{\frac{1}{2}b^2c^2y^2z\dot{z}}{a + cx}$$

= the fluxion of the area ASM. In order to find the

fluent

fluent hereof, let $\frac{1}{a+cx}$ be resolved into the series $\frac{1}{a} - \frac{2cx}{a^2}$, &c. (whereof the two first terms will, here, be sufficient) by which means our fluxion will become $\frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{a^2} - \frac{b^2c^2x\dot{z}y^2}{a^3}$; which, because $y\dot{z} = -\dot{x}$, and $x\dot{z} = \dot{y}$, will be reduced to $\frac{1}{2}b^2\dot{z} + \frac{b^2c^2}{2a^2} \times -\dot{x}y - \frac{b^2c^2y^2\dot{y}}{a^3}$; whose fluent will therefore be truly expressed by $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times \text{area DHE} - \frac{b^2c^2y^3}{3a^3}$, or $\frac{1}{2}b^2z + \frac{b^2c^2}{2a^2} \times \frac{1}{2}z - \frac{1}{2}xy - \frac{b^2c^2y^3}{3a^3}$ (because the area DHE = $\frac{1}{2}$ DE $\times$ DF = $\frac{1}{2}$ HF $\times$ EH = $\frac{1}{2}z - \frac{1}{2}xy$). This expression, when M coincides with P, and z is = the semi-circumference DEK (= p), will become $\frac{b^2p}{2} + \frac{b^2c^2p}{4aa} = \frac{b^2dp}{2}$ (supposing $d = 1 + \frac{cc}{2aa}$) = the area of the semi-ellipsis ABP. Therefore we have, as $\frac{b^2dp}{2} : \frac{b^2dz}{2} = \frac{b^2c^2xy}{4aa} - \frac{b^2c^2y^3}{3a^3}$ (= the area ASM) :: p (= the length of an arch of 180°) : $z - \frac{c^2xy}{2a^2d} - \frac{2c^2y^3}{3a^3d}$ = the length of an arch (A) expressing the planet's mean anomaly : from which equation, $z = A + \frac{c^2xy}{2a^2d} + \frac{2c^2y^3}{3a^3d} = A + \frac{c^2}{4a^2d} \times \overline{\text{fin. } 2z} + \frac{2c^3}{3a^3d} \times \overline{\text{fin. } z}$ (because xy , or $\text{co-fin. } z \times \text{fin. } z = \frac{1}{2} \text{fin. } 2z$): where the two last terms being very small in comparison of the others (and, therefore, z nearly = A), we may, instead of $\text{fin. } z$ and $\text{fin. } 2z$, substitute $\text{fin. } A$ and $\text{fin. } 2A$; by which means we have $z = A + \frac{cc}{4a^2d} \times \text{fin. } 2A + \frac{2c^3}{3a^3d} \times \overline{\text{fin. } A}$. From whence it appears, that, in order to have the angle AFM at the upper focus, the mean anomaly (A) of the planet at the time given, must be increased by the quantity, or correction $\frac{cc}{4a^2d} \times \text{fin. } 2A + \frac{2c^3}{3a^3d} \times \overline{\text{fin. } A}$.

But to express the value of this correction in seconds of a degree (which in practice is the most commodious) it will

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will be, as 3,1416 (the length of an arch of 180 degrees) is to 648000 (the number of seconds in that arch) so is $\frac{cc}{4a^2d} \times \sin. 2A + \frac{2c^3}{3a^3d} \times \overline{\sin. A}^3$ to $51567 \times \frac{c^3}{a^2d} \times \sin. 2A + 137513 \times \frac{c^3}{a^3d} \times \overline{\sin. A}^3$ = the number of seconds in the said correction: the logarithm of the latter term of which will, therefore, be = $5,1383 - \log. d + 3 \log. \frac{c}{a} + 3 \log. \sin. A$; and that of the former = $4,7123 - \log. d + 2 \log. \frac{c}{a} + \log. \sin. 2A$. But, to render these expressions still more convenient for practice, the log. of d , by reason of its smallness, may be, either, intirely neglected, or else so assumed, to be nearly a mean of what it is known to be in the planetary orbits. Accordingly, by assuming the excentricity $c = \frac{8}{100}$ of the mean distance (which is a small matter less than the excentricity of *Mars*, but something greater than *those* of the *Moon*, *Saturn*, and *Jupiter*), the value of $d (= 1 + \frac{cc}{2aa})$ will be = 1,0032, and its logarithm = 0,0014. Whence the log. of the former part of our correction, by substituting this value, will be $5,1369 + 3 \log. \frac{c}{a} + 3 \log. \sin. A = 3 \times 1,7123 + \log. \frac{c}{a} + \log. \sin. A$; and that of the latter = $4,7109 + 2 \log. \frac{c}{a} + \log. \sin. 2A$; which, expressed in words at length, give the following

Practical Rules.

1°. To the sum of the constant logarithm 1,7123 and the log. of the excentricity in parts of the mean distance, add the log. sine of the mean anomaly; the sum (rejecting the radius) being tripled, will give the log. of the first equation (in seconds) to be added to the mean anomaly.

2°. To the sum of the constant log. 4,7109 and twice the log. of the excentricity, add the log. sine of twice the mean anomaly; the sum (rejecting the radius) will be the log. of the second equation; to be added or subtracted, according as the mean anomaly is less, or greater than 90 degrees.

Here

Here, and in what follows, the anomaly is to be always reckoned from, or to the aphelion, the nearest way; in which the seconds may be omitted, in computing the proposed corrections. Which corrections being made, the true anomaly, or angle at the lower focus, will be had from the common proportion, by saying, as the aphelion-distance is to the perihelion-distance, so is the tangent of half the mean anomaly, thus corrected, to the tangent of half the true anomaly sought.

As an example of the use of the method here laid down, let the excentricity be supposed = 0,048219 (being that assigned, by Dr. HALLEY, to the orbit of Jupiter) and let the mean anomaly be 45° .

Then, log. excent. . .	<u>2,6832</u>	2 log. excent. . . .	3,3664
+ const. log. . .	<u>1,7123</u>	+ const. log. . . .	4,7109
= log. for 1st equ.	0,3955	= log. for 2d equ.	<u>2,0773</u>
log. fin. anom. . .	<u>9,8494</u>	log. fin. 2 anom.	<u>10,0000</u>
	0,2449	2d equ. = $119\frac{1}{2}''$	<u>2,0773</u>
first equ. = $5\frac{1}{2}''$	0,7347		

From 1,978537 = log. of perihelion-dist. 0,951781,

subtr. 0,020452 = log. of the aphelion-dist. 1,048219;

therem. 1,958085, will be a (3d) const. log. for this orbit: to which

add 9,617596 = log. tang. $\frac{1}{2}$ cor. anom. $22^{\circ} 31' 2\frac{1}{2}''$,

so shall 9,575681 = log. tang. $\frac{1}{2}$ true anomaly $20^{\circ} 37' 40''$:

whose double, $41^{\circ} 15' 20''$, is therefore the true anomaly required.

The same excentricity being retained; let the mean anomaly be, now, supposed 120 degrees.

Then, log. fin. anom.	9,9375	log. fin. 2 anom. . .	9,9375
log. for first equat.	<u>0,3955</u>	log. for 2d equation	<u>2,0773</u>
	0,3330	2d equat. = $103\frac{1}{2}''$	<u>2,0148</u>
first equat. = $10''$	0,9990		

Hence $120^{\circ} + 10'' - 1' 43\frac{1}{2}'' = 119^{\circ} 58' 26\frac{1}{2}'' =$ the cor. anomaly.

Therf. log. tang. $\frac{1}{2}$ cor. anom. $59^{\circ} 59' 13\frac{1}{2}''$. . . 10,238331

+ third const. log. for this orbit 1,958085

= log. tang. $\frac{1}{2}$ true anom. $57^{\circ} 32' 10''$ 10,196416

H

Whence

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Whence the anomaly itself is given $= 3^{\circ} 25' 4'' 20'''$.

If the excentricity be assumed $= 0,066777$ (being the greatest that Dr. HALLEY gives to the lunar orbit) the two constant logarithms, to be added to the sines of the mean anomaly and of its double, will be $0,5369$ and $2,3601$: whence if the said anomaly be taken $= 50^{\circ}$ (at which, according to the *Doctor's Table*, *pro expediendo calculo æquationis centri lunæ*, the whole correction is a *maximum*), the former part of the said correction will be found $= 18\frac{1}{2}''$, and the latter part $= 225\frac{1}{2}''$: therefore the sum of both is $244''$, or $4' 4''$; agreeing, exactly, with the quantity given in the *Table*. And in the very same manner, the proper corrections corresponding to other anomalies and excentricities may be computed; the error never amounting to above a single second in any of the planets, except *Mars* and *Mercury*: in the place of *Mars*, the greatest error will be two or three seconds; and in that of *Mercury*, about as many minutes.—As to the *Earth* and *Venus*, the second equation, alone, will be sufficient to give their places to less than a second.

To obtain a farther correction, which will be necessary when the orbit is very eccentric, we may (instead of the two first only) make use of a greater number of terms of the series $\frac{1}{a^2} - \frac{2cx}{a^3} + \frac{3c^2x^2}{a^4} - \frac{4c^3x^3}{a^5} \&c. (= \frac{1}{a+cx})$; by

which means the fluxion $(\frac{1}{2}b^2\dot{z} + \frac{\frac{1}{2}b^2c^2y^2\dot{z}}{a+cx})$ of the area MSA

described about the focus S, which is proportional to the time, will be here represented by

$\frac{1}{2}b^2\dot{z} + \frac{1}{2}b^2 \times \frac{e^2y^2\dot{z}}{a^2} - 2e^3y^2x\dot{z} + 3e^4y^2x^2\dot{z} - 4e^5y^2x^3\dot{z} \&c. (supposing e = \frac{c}{a})$. But it is well known that $y^2 (\sin. z)^2 = \frac{1}{2} - \frac{1}{2} \cosin. 2z$; whence $y^2x (= y^2 \times \cosin. z) = \frac{1}{2} \cosin. z - \frac{1}{2} \cosin. 2z \times \cosin. z = \frac{1}{2} \cosin. z - \frac{1}{4} \cosin. z - \frac{1}{4} \cosin. 3z = \frac{1}{4} \cosin. z - \frac{1}{4} \cosin. 3z$; and therefore y^2x^2

* This, and all that follows to the same purpose, is nothing more than the application of the THEOREM, That the rectangle of the co-sines of any two angles (the radius being unity) is equal to half the sum of the co-sines of the sum and difference of those angles.

$= \frac{1}{4} \text{co-fin. } z \times \text{co-fin. } z - \frac{1}{4} \text{co-fin. } 3z \times \text{co-fin. } z = \frac{1}{8} + \frac{1}{8} \text{co-fin. } 2z - \frac{1}{8} \text{co-fin. } 2z - \frac{1}{8} \text{co-fin. } 4z = \frac{1}{8} - \frac{1}{8} \text{co-fin. } 4z$: whence also $y^2 x^3 = \frac{1}{8} \text{co-fin. } z - \frac{1}{8} \text{co-fin. } 4z \times \text{co-fin. } z = \frac{1}{8} \text{co-fin. } z - \frac{1}{16} \text{co-fin. } 3z - \frac{1}{16} \text{co-fin. } 5z$. Which values being now substituted for their equals, our fluxion, above given, will become

$\frac{1}{2} b^2 \dot{z} + \frac{1}{2} b^2$ into $e^2 \times \frac{1}{2} \dot{z} - \frac{1}{4} \times 2 \dot{z} \text{co-fin. } 2z - 2e^3 \times \frac{1}{4} \dot{z} \text{co-f. } z - \frac{1}{12} \times 3 \dot{z} \text{co-f. } 3z + 3e^4 \times \frac{1}{8} \dot{z} - \frac{1}{12} \times 4 \dot{z} \text{co-f. } 4z - 4e^5 \times \frac{1}{8} \dot{z} \text{co-f. } z - \frac{1}{8} \times 3 \dot{z} \text{co-f. } 3z - \frac{1}{80} \times 5 \dot{z} \text{co-f. } 5z + \mathcal{E}c$. and, consequently, the fluent thereof $= \frac{1}{2} b^2 z + \frac{1}{2} b^2$ into $e^2 \times \frac{1}{2} z - \frac{1}{4} \text{fin. } 2z - 2e^3 \times \frac{1}{4} \text{f. } z - \frac{1}{12} \text{f. } 3z + 3e^4 \times \frac{1}{8} z - \frac{1}{12} \text{fin. } 4z - 4e^5 \times \frac{1}{8} \text{fine } z - \frac{1}{8} \text{fine } 3z - \frac{1}{80} \text{fine } 5z + \mathcal{E}c$. $= \frac{1}{2} b^2$ into $1 + \frac{1}{2} e^2 + \frac{1}{8} e^4 \times z - \frac{1}{2} e^3 + \frac{1}{2} e^5 \times \text{fin. } z - \frac{1}{4} e^2 \text{fin. } 2z + \frac{1}{6} e^3 + \frac{1}{12} e^5 \times \text{fin. } 3z - \frac{3e^4}{32} \text{fin. } 4z + \frac{e^5}{20} \text{fin. } 5z$ (supposing all

such terms wherein e rises to more than five dimensions, to be disregarded). This expression, when $z = p$ ($=$ the semi-circumference AEK) becomes $\frac{1}{2} b^2 \times 1 + \frac{1}{2} e^2 + \frac{1}{8} e^4 + \mathcal{E}c. \times p = \frac{1}{2} b^2 p \times \sqrt{1 - ee}^{-\frac{1}{2}} = \frac{\frac{1}{2} b^2 p}{\sqrt{1 - \frac{ee}{aa}}} = \frac{\frac{1}{2} b^2 p \times a}{\sqrt{a^2 - c^2}} = \frac{1}{2} abp =$ the area

of the semi-ellipse ABP. Hence it will be, as $\frac{1}{2} abp$ (area ABP) : $\frac{1}{2} abz - \frac{1}{2} b^2 \times \frac{1}{2} e^3 + \frac{1}{2} e^5 \times \text{fin. } z - \frac{1}{2} b^2 \times \frac{1}{4} e^2 \text{fin. } 2z + \mathcal{E}c$. (area ASM) :: p (the length of an arch of 180°) : $z - \frac{b}{a} \times \frac{1}{2} e^3 + \frac{1}{2} e^5 \times \text{fin. } z - \frac{b}{a} \times \frac{e^2}{4} \text{fin. } 2z + \mathcal{E}c$. $=$ the length of the arch A , expressing the planet's mean anomaly: whence, by substituting $f = \frac{b}{a}$, we have $A = z - \frac{1}{1 + e^2 \times \frac{1}{2} f e^3 \text{fin. } z - \frac{1}{4} f e^2 \text{fin. } 2z + 1 + \frac{1}{2} e^2 \times \frac{1}{6} f e^3 \text{fin. } 3z - \frac{3f e^4}{32} \times \text{fin. } 4z + \frac{f e^5}{20} \times \text{fine } 5z$.

Now, to find from hence the value of z , in terms of A and the sines of its multiples, we may, for a first approximation, assume $1 + e^2 \times \frac{1}{2} f e^3 \text{fin. } z + \frac{1}{4} f e^2 \text{fin. } 2z + \mathcal{E}c$. as equal to $1 + e^2 \times \frac{1}{2} f e^3 \text{fin. } A + \frac{1}{4} f e^2 \text{fin. } 2A - 1 + \frac{1}{2} e^2 \times \frac{1}{6} f e^3 \text{fin. } 3A + \mathcal{E}c$. which last quantity being denoted by Q , we shall have $A = z - Q$, and consequently $z = A + Q$.

But $\sin. z (= \sin. \overline{A + Q}) = \sin. A \times \text{co-sin. } Q + \sin. Q \times \text{co-sin. } A = \sin. A + Q \times \text{co-sin. } A$, nearly (because Q being very small, its co-sine will differ insensibly from the radius, and the sine very little from the arch itself). In the same manner, $f. 2z (= f. 2A + 2Q) = f. 2A + 2Q \times \text{co-f. } 2A$, $\sin. 3z = f. 3A + 3Q \times \text{co-f. } 3A$, &c. Which values being therefore substituted in the given equation, it will become $A = z - Q - Q \times \frac{1 + e^2 \times \frac{1}{2} f e^3 \text{co-f. } A + \frac{1}{2} f e^3 \text{co-f. } 2A - 1 + \frac{1}{2} e^2 \times \frac{1}{2} f e^3 \text{co-f. } 3A \text{ \&c.}}{1 + e^2 \times \frac{1}{2} f e^3 \text{co-sin. } A}$. But the first term of the series, $\frac{1}{1 + e^2 \times \frac{1}{2} f e^3 \text{co-sin. } A}$, drawn into $-Q$ (if all terms having more than five dimensions of e be neglected) will be barely $= \frac{1}{2} f e^3 \times \text{co-sin. } A \times -\frac{1}{2} f e^3 \sin. 2A = -\frac{1}{8} f^2 e^5 \times \text{co-sin. } A \times \sin. 2A = -\frac{1}{16} f^2 e^5 \times \sin. 3A + \sin. A$.

In the same manner $\frac{1}{2} f e^2 \text{co-sin. } 2A \times -Q = -\frac{1}{2} f e^2 \text{co-sin. } 2A \times \frac{1}{2} f e^3 \sin. A + \frac{1}{4} f e^2 \sin. 2A - \frac{1}{6} f e^3 \sin. 3A = -\frac{1}{8} f^2 e^5 \times \sin. 3A - \sin. A - \frac{1}{16} f^2 e^4 \times \sin. 4A + \frac{1}{24} f^2 e^5 \times \sin. 5A + \sin. A$. And, lastly, $-1 + \frac{1}{2} e^2 \times \frac{1}{2} f e^3 \text{co-sin. } 3A \times -Q$ equal to $\frac{1}{2} f e^3 \text{co-sin. } 3A \times Q = \frac{1}{2} f e^3 \times \text{co-sin. } 3A \times \frac{1}{4} f e^2 \sin. 2A = \frac{1}{16} f^2 e^5 \times \sin. 5A - \sin. A$. The sum of all which will be $\frac{1}{24} f^2 e^5 \sin. A - \frac{1}{16} f^2 e^5 \sin. 3A - \frac{1}{16} f^2 e^4 \sin. 4A + \frac{5}{48} f^2 e^5 \sin. 5A$; which added to $z - Q$ (or its equal, $z - 1 + e^2 \times \frac{1}{2} f e^3 \sin. A - \frac{1}{4} f e^2 \sin. 2A$, &c.), gives $A = z - 1 + e^2 - \frac{1}{12} f e^2 \times \frac{1}{2} e^3 f \sin. A - \frac{1}{4} e^2 f \sin. 2A + 1 + \frac{1}{2} e^2 - \frac{3}{8} e^4 f \times \frac{1}{6} e^3 f \sin. 3A - \frac{3f + 2f^2 \times \frac{e^4}{32} \sin. 4A + \frac{1}{24} f + \frac{5}{48} f^2 e^5 \times \sin. 5A}{12}$. From whence

we have $z = A + 1 + \frac{11e^2}{12} \times \frac{1}{2} e^3 f \sin. A + \frac{1}{4} e^2 f \sin. 2A - 1 - \frac{5e^2}{8} \times \frac{1}{6} e^3 f \sin. 3A + \frac{5e^4 f^2}{32} \sin. 4A - \frac{37e^5 f^3}{240} \sin. 5A$, very near;

because in all terms having more than three dimensions of e , the quantities 1 and f may be used indifferently, for each other, without producing any errors but such as consist of more than five dimensions of the converging quantity e .

But, since it is known that $\sin. 3A = 3S - 3S^3$, and $\sin. 5A = 5S - 20S^3 + 16S^5$ (S being the sine of A) we shall, by substituting these values in the 2d, 4th, and 6th terms (after proper

proper reduction) have $z = A + \overline{1 + 4ee \times \frac{2}{3} e^3 f S^3 - \frac{37e^3 f S^5}{15}}$

$+ \frac{1}{4} e^2 f \times \sin. 2A + \frac{5e^4 f^2}{32} \times \sin. 4A$. Hence it appears that

$$\frac{180 \times 60 \times 60}{3,1416} \times \overline{1 + 4ee \times \frac{2}{3} e^3 f S^3 - \frac{37e^3 f S^5}{15}} + \frac{1}{4} e^2 f \sin. 2A + \frac{5e^4 f^2}{32} \sin. 4A$$

will express the number of seconds, to be added to the mean anomaly, in order to have the angle AFM at the upper focus of the ellipsis, corresponding to that anomaly. From whence is deduced the following method of calculation.

Let F denote the logarithm of half the lesser axis divided by half the greater, E the log. of the eccentricity divided by half the greater axis, G the log. of the sum of the squares of half the greater axis and twice the eccentricity divided by the square of half the greater axis.

$$\text{Take } P = 1,71277 + E + \frac{1}{3}F + \frac{1}{3}G,$$

$$Q = 1,14130 + E + \frac{1}{5}F,$$

$$R = 4,71236 + 2E + F,$$

$$S = 4,50824 + 4E + 2F;$$

then the logarithms of four equations (in seconds), to be applied to the mean anomaly (A), will be

$$3 \times \overline{P + \log. \sin. A - \log. \text{rad.}}$$

$$5 \times \overline{Q + \log. \sin. A - \log. \text{rad.}}$$

$$R + \log. \sin. 2A - \log. \text{rad.}$$

$$S + \log. \sin. 4A - \log. \text{rad.}$$

Of which equations the first is always to be added, and the second always subtracted; the other two being to be added, or subtracted, according as the sines of their respective arguments, $2A$ and $4A$, are positive, or negative.

The two principal of these equations agree with those before given; and are the same, in effect, with the two equations laid down (without demonstration) by Sir ISAAC NEWTON, in the Scholium to the 31st Proposition of the first book of his *Principia*. The latter of which, in the *Laws of the Moon's Motion*, prefixed to that Work, seems to be represented, as defective; it being there asserted, that, *the inequality in the motion of a planet about the upper focus, consists of three parts*; as if the nature of the subject admitted of just that number, and no more.

more; whereas the parts, or equations arising in the consideration, are without number, being the terms of an infinite series, wherein the eccentricity is the converging quantity. Sir ISAAC NEWTON has given two terms of this series, which are right; but the new equation added by his *Commentator*, is not so; the sign thereof, the coefficient, and the law by which it increases and decreases, being all different from what they ought to be.

This equation (expressed according to the above notation, where i represents half the greater axis, f half the lesser axis, e the eccentricity, and A the mean anomaly) he makes to be $-\frac{1}{2}e^4 \times \overline{\sin. A}^3 \times \text{co-sin. } A$. But it ought to be $+\frac{3f+2f^2}{3^2} \times e^4 \sin. 4A$ ($= \frac{5e^4}{3^2} \times \sin. 4A$, nearly), as is shewn above; this being the 3d term of the general series, and the next in order after those given by Sir ISAAC NEWTON; who appears, more than once*, to have been disadvantageously (I might say, unfairly) represented, and that, under the covers of his own book: a circumstance that cannot be attended to without some concern and dislike, by those who entertain a due regard for the merit of an Author to whom the mathematical world is so much indebted.

I shall now put down one single example of the use of the equations above derived; wherein I shall suppose the eccentricity to be $\frac{2}{1000000}$ parts of the semi-transverse axis (the same as is assigned by Dr. HALLEY to the orbit of Mercury). Here, then, we have $E = -1,313635$; $F (= \log. \sqrt{1-ee} = \frac{1}{2} \log. 1-ee) = -0,009406$; $G (= \log. 1+4ee) = 0,068024$; whence $P = 1,046$; $Q = 0,453$; $R = 3,3302$; $S = 1,744$:

* In the 28th Proposition of his third book, it is found that the moon's distance from the earth in the syzgies is to the distance in the quadratures (setting aside the consideration of eccentricity) as 69 to 70; which is confirmed by what is demonstrated in a subsequent part of this our Work, as well as by the calculations of others; nevertheless the truth of this proportion is called in question, and a new one is laid down, which makes the said distances to be in the proportion of 59 to 60. See *Laws of the Moon's Motion*, p. 11 and 12.
which

which values will serve in all cases belonging to this orbit. Whence, supposing the given anomaly to be 50° , we have

1,0460	0,4530	3,3302	1,744
<u>9,8842</u>	<u>9,8842</u>	<u>9,9933</u>	<u>9,534</u>
0,9302	0,3372	3,3235	1,278.
<u>3</u>	<u>5</u>		
2,7906	1,6860		

Of which resulting logarithms, the numbers corresponding are $617\frac{1}{2}$, $48\frac{1}{2}$, $2106\frac{1}{2}$, and 19; whereof the first and third being added, and the other two subtracted, we have $50^\circ 44' 16''$ for the corrected anomaly, or the angle at the upper focus; whereof the half is $25^\circ 22' 8''$:

$$\begin{aligned} &\text{Therefore log. tang. } 25^\circ 22' 8'' \dots\dots\dots 9,6759338 \\ &+ \text{log. of the ratio of the gr. and least dist. } 1,8185730 \\ &= \text{log. tang. } \frac{1}{2} \text{ true anomaly } 17^\circ 20' 28'' \quad 9,4945068. \end{aligned}$$

Which conclusion is true to a second. Nor will the error, in any part of this orbit, amount to more than about two or three seconds.—If you would have the result depended on to a single second, or if the orbit be supposed still more eccentric than that of Mercury, then the following method may be of use.

Say, as half the greater axis of the ellipsis is to the eccentricity, so is the sine of the mean anomaly to the sine of an angle; which subtract from the mean anomaly, and to the log. sine of the remainder (which I call the eccentric anomaly) add the sum of the log. of the eccentricity and the constant log. 1,758123: the aggregate (rejecting the radius) will be the logarithm of an angle, in degrees and decimal parts; which, subtracted from the angle first found, leaves a correction to be added (under its proper sign) to the mean anomaly: with which corrected anomaly, let the whole operation be repeated, if needful, by always adding the last correction to the mean anomaly. Then it will be, as the greater semi-axis of the ellipsis is to the lesser, so is the tangent of the corrected anomaly to the tangent of the angle at the upper focus of the ellipsis: whence the angle at the lower focus, or the true anomaly, may also

also be known by the common proportion, and that to any assigned degree of exactness.

This method, in all the planets, except *Mars* and *Mercury*, answers to a second, at one operation. In the former of these two, the error, when greatest, will amount to about three or four seconds; and in the latter, to nearly as many minutes; in which case, three operations will be necessary: but in order to avoid that trouble, the following calculation may be used; which is so exact as to answer, even in the orbit of *Mercury*, to less than half a second, without repeating the operation.

Add together twice the log. of the eccentricity, the log. secant of the angle first found (as above), and the log. co-sine of the sum of the eccentric anomaly and mean anomaly once corrected (in all of which angles the seconds may be neglected). The aggregate (subtracting twice the radius) will be the log. of a fraction to be added to unity, when the said sum of anomalies is between 90 and 270° ; but otherwise, subtracted therefrom: then the log. of this sum, or remainder being subtracted from the log. of the first correction, you will have the log. of the true correction to be added (under its proper sign) to the mean anomaly given.

Thus, for example, let the mean anomaly be 70 ; and let the eccentricity be $= 0,20589$ (as in the preceding example).

$$\begin{array}{rcl} \text{Here, fin. mean anom. } 70^\circ & \dots\dots\dots & 9,9729858 \\ \text{log. eccent. } & \dots\dots\dots & 1,3136353 \\ \therefore \text{ ang. first found } = 11^\circ 9',33 & & 9,2866211; \end{array}$$

whence $58^\circ 50',67 =$ the eccentric anomaly :

$$\begin{array}{rcl} \text{whose sine } & \dots\dots\dots & 9,9323552 \\ + \text{ the log. proper for the orbit } & \dots\dots\dots & 1,0717583 \\ = \text{ log. of } 10^\circ,0952 & \dots\dots\dots & 1,0041135: \end{array}$$

which angle, or its equal, $10^\circ 5',71$, being subtracted from $11^\circ 9',33$, leaves the first correction $1^\circ 3',62 = 63',62$.

Q. E. D.

Moreover,

Moreover, by adding together 70° , $1^{\circ} 3'$, and $58^{\circ} 50'$, we have

$$\begin{array}{rcl} 129^{\circ} 53', \text{ whose co-sine} & & 9,8070 \\ + \text{sec. } 11^{\circ} 9' \text{ ang. first found} & . . . & 10,0083 \\ + \text{twice log. eccent.} & & \underline{2,6272} \\ = \text{log. of } 0,0272 & & 2,4425 \end{array}$$

$$\begin{array}{rcl} \text{Log. first cor. } 63',62 & & 1,80359 \\ - \text{log. } 1,0272 & & \underline{0,01187} \\ = \text{log. true cor. } 61',9 & . . . & 1,79172 \end{array}$$

$$\begin{array}{rcl} \text{Now the mean anom.} + \text{true cor.} = 71^{\circ} 1',9, & \} & 10,4638087 \\ \text{whose log. tang.} & & \\ + \text{log. of the semi-conjugate axis} & & \underline{1,9905940} \\ = \text{log. tang. of } 70^{\circ} 38',82 & & 10,4544027 \end{array}$$

$$\begin{array}{rcl} \text{And the log. tang. of the half hereof, } 35^{\circ} 19',41 & & 9,8504350 \\ + \text{log. of the ratio of the gr. and least distances} & & \underline{1,8185730} \\ = \text{log. tang. of } 25^{\circ} 1',02 & & 9,6690080 \end{array}$$

whose double $50^{\circ} 2',04$, or $50^{\circ} 2' 2''$, is the true anomaly required.





A

DETERMINATION

OF THE

Difference between the MOTION of a COMET in an Elliptic, and a Parabolic Orbit.

Fig. 19.  Let PNG be a parabola, and PBH a very excentric ellipsis, having the same focus S, and vertex P with the parabola; let moreover N and n be considered as cotemporary positions of two bodies, in these orbits, moving from the perihelion P at the same time, about the sun in the focus S. Make NBC perpendicular to PSO, and call PC, x ; PS, c ; and the greater axis of the ellipsis, a : then the lesser axis will be $= 2\sqrt{c \times a - c^2}$; the parameter $= \frac{4ac - 4c^2}{a}$; and the ordinate BC $= 2\sqrt{\frac{ac - cc \times ax - xx}{aa}}$ (by the property of the ellipsis) $= 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \sqrt{1 - \frac{c}{a}} \times \sqrt{1 - \frac{x}{a}} = 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \sqrt{1 - \frac{c}{2a} - \frac{x}{2a}}$, nearly. This last taken from CN ($= 2c^{\frac{1}{2}}x^{\frac{1}{2}}$, by the nature of the parabola) leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{c}{a} + \frac{x}{a} = BN$; which being multiplied by x and the fluent found, we thence have $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{2cx}{3a} + \frac{2x^2}{5a}$ for the measure of the area NPB, or NPv, very near: which subtracted from the area NSP ($= CN \times \frac{2}{3}PC - CN \times \frac{1}{2}CS = 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{2}{3}x - 2c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{1}{2}x - \frac{1}{2}c = c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x$), leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x - \frac{2cx}{3a} - \frac{2xx}{5a} =$ the area vPS. Moreover, the parameter of the parabola being $4c$, and that

that of the ellipsis $= \frac{4ac - 4cc}{a}$, we have $\sqrt{4c}$ to $\sqrt{\frac{4ac - 4cc}{a}}$, so is the parabolic area NSP ($c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x$) to the corresponding, or cotemporary elliptical area nSP $= \sqrt{1 - \frac{c}{a} \times c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x}$
 $= c^{\frac{1}{2}}x^{\frac{1}{2}} \times 1 - \frac{c}{2a} \times c + \frac{1}{3}x$ (nearly) $= c^{\frac{1}{2}}x^{\frac{1}{2}} \times c + \frac{1}{3}x - \frac{cc}{2a} - \frac{cx}{6a}$;
 which subducted from the area vPS, leaves $c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{cc}{2a} - \frac{cx}{2a} - \frac{2xz}{5a}$
 $=$ the area vSn.

Let now NM be supposed perpendicular to the tangent NA, meeting the axis in M; then SM, SN, and SA will be all equal to each other, by the nature of the parabola; and consequently the angle PMN equal to half PSN; whose tangent (to the radius 1) let be denoted by z : whence it will be, as $1 : z :: MC (2c) : CN (2c^{\frac{1}{2}}x^{\frac{1}{2}})$; and consequently $x = cz^2$. Which value being substituted in the area vSn, it will become

$= cz \times \frac{cc}{2a} - \frac{c^2z^2}{2a} - \frac{2c^2z^4}{5a} = \frac{c^3z}{2a} \times 1 - z^2 - \frac{4}{5}z^4$; and this di-

vided by $\frac{1}{2} \overline{Sv}^2 (= \frac{1}{2} \times c + x = \frac{1}{2}c^2 \times 1 + z^2)$, gives $\frac{c^3z}{2a} \times \frac{1 - z^2 - \frac{4}{5}z^4}{1 + z^2}$ for the measure of the angle vSn, in parts of the radius (1). Therefore, if m be put $= 3437$ = the number of minutes in an arch equal to the radius, and $u = \frac{z \times 1 - z^2 - \frac{4}{5}z^4}{1 + z^2}$,

it is evident that $\frac{c}{a} \times mu$ will express the measure of the said angle in minutes of a degree. To find now the ratio Sn to SN, which still remains to be determined, we have (by Tri-

gonometry) $Nv = \frac{\text{fin. B}}{\text{fin. Bvf}} \times BN = \frac{\text{fin. AFC}}{\text{fin. AFS}} \times BN = \frac{\text{fin. M}}{\text{co-fin. M}} \times BN = \text{tang. M} \times BN = z \times c^{\frac{1}{2}}x^{\frac{1}{2}} \times \frac{c + x}{a} = z \times cz \times \frac{c + cz^2}{a}$
 $= \frac{cczz}{a} \times 1 + z^2$. Also (supposing nb perpendicular to Sv) it

will be, as 1 (radius) : $\frac{cu}{a}$ (the arch measuring the angle nSv)

$:: c \times \overline{1 + zz} (= Sn, \text{ nearly}) : nb = \frac{ccu}{a} \times \overline{1 + zz} :$ whence,
 again, by fimilar triangles, $1 : z (:: CM : CN) :: \frac{ccu}{a} \times \overline{1 + zz}$
 $(nb) : bv = \frac{cczu}{a} \times \overline{1 + zz} :$ this added to $Nv = \frac{cczz}{a} \times \overline{1 + zz},$
 gives $Nb = \frac{ccz}{a} \times \overline{u + z} \times \overline{1 + zz} :$ which therefore is to SN
 $(c \times \overline{1 + zz})$ as $\frac{cz}{a} \times \overline{u + z}$ to unity; and consequently the re-
 quired proportion of Sn (or Sb) to SN, as $1 - \frac{cz}{a} \times \overline{u + z}$ to
 unity.

From this last conclusion, and that derived above, exhibiting the value of the angle NSn, the following Table is computed; whose use is thus: Find, in the first column, the comet's longitude from the perihelion, as given from the hypothesis of a parabolic orbit (either by Dr. HALLEY's Table, or any other of the like kind); against which, in the third column, you have the logarithm of a number of minutes (expressed in the second column); from which subtracting the logarithm of the ratio of the greater axe of the ellipse divided by the perihelion distance, the remainder will be the logarithm of a number of minutes to be added to, or subtracted from the aforesaid longitude, as the Table directs: whereby the comet's longitude, for the same time, in the elliptic orbit will be given. And if, from the logarithm found in the fourth column, the logarithm of the same ratio be also subtracted, the remainder, abating 10, will be the logarithm of a quantity to be taken from the logarithm of the comet's distance from the sun, computed according to the aforesaid hypothesis.

Thus, for example, let the greater axis of the ellipsis be supposed = 35,727, and the perihelion distance = 0,5825 (answering to the orbit of the comet of the year eighty-two); and let the longitude from the perihelion, according to Dr. HALLEY's Table, computed for a parabolic orbit, be $44^{\circ} 3' 20''$; corresponding to which, the logarithm of the distance from the sun

fun is given $= 0,065838$. Here, then, the log. of $\frac{35,727}{0,5825}$ being $= 1,7877$, this value is to be subtracted from both the logarithms $2,9228$ and $9,0555$, standing against 44° , in the third and fourth columns of the annexed Table: from whence will be found the two quantities $13',65$ and $0,001853$; which being subtracted from $44^\circ 3' 20''$, and $0,065838$, the required longitude from the perihelion is given from thence $= 43^\circ 49' 41''$, and the logarithm of the comet's distance from the fun $= 0,063985$.

The same Table, not only furnishes an easy way, for deducing the motion in an elliptic orbit, from the motion in a parabolic one, but may be farther useful in determining, in some degree, the species of the ellipsis which a new comet describes, when the observations thereon are found to differ sensibly from the places computed according to the hypothesis of a parabolic orbit.

Of the Difference of Motion of a Comet

	subtr.	subtract	subtract		subtr.	subtract	subtract
1°	30	1,4770	5,8200	41°	836	2,9220	9,0006
2	60	1,7770	6,4220	42	836	2,9229	9,0195
3	90	1,9533	6,7735	43	836	2,9232	9,0378
4	119	2,0778	7,0242	44	836	2,9228	9,0555
5	149	2,1740	7,2178	45	835	2,9217	9,0727
6	179	2,2521	7,2759	46	832	2,9200	9,0894
7	208	2,3178	7,5095	47	827	2,9176	9,1056
8	237	2,3745	7,6250	48	821	2,9145	9,1215
9	266	2,4242	7,7269	49	814	2,9107	9,1369
10	294	2,4683	7,8178	50	806	2,9061	9,1520
11	322	2,5074	7,8998	51	796	2,9007	9,1666
12	349	2,5431	7,9747	52	784	2,8945	9,1808
13	376	2,5760	8,0437	53	772	2,8874	9,1947
14	403	2,6058	8,1074	54	758	2,8795	9,2083
15	430	2,6331	8,1666	55	743	2,8708	9,2215
16	455	2,6583	8,2217	56	726	2,8610	9,2344
17	480	2,6816	8,2735	57	708	2,8500	9,2469
18	505	2,7032	8,3222	58	688	2,8378	9,2592
19	529	2,7233	8,3682	59	667	2,8243	9,2712
20	552	2,7420	8,4116	60	645	2,8095	9,2829
21	575	2,7594	8,4528	61	621	2,7932	9,2943
22	599	2,7755	8,4921	62	596	2,7754	9,3055
23	617	2,7906	8,5295	63	570	2,7557	9,3163
24	637	2,8046	8,5652	64	542	2,7338	9,3270
25	657	2,8176	8,5994	65	512	2,7095	9,3374
26	676	2,8297	8,6320	66	482	2,6827	9,3477
27	694	2,8410	8,6634	67	450	2,6526	9,3576
28	710	2,8514	8,6934	68	416	2,6187	9,3674
29	726	2,8610	8,7224	69	381	2,5805	9,3769
30	741	2,8698	8,7502	70	344	2,5363	9,3862
31	755	2,8779	8,7770	71	306	2,4857	9,3954
32	768	2,8853	8,8030	72	267	2,4258	9,4044
33	780	2,8920	8,8280	73	226	2,3537	9,4132
34	791	2,8980	8,8521	74	184	2,2639	9,4218
35	801	2,9034	8,8755	75	140	2,1458	9,4303
36	809	2,9081	8,8980	76	95	1,9707	9,4386
37	817	2,9122	8,9198	77	48	1,6840	9,4468
38	823	2,9156	8,9409	78	add	add	9,4546
39	829	2,9184	8,9614	79	49	1,6910	9,4624
40	833	2,9205	8,9813	80	100	2,0006	9,4701

	add	add	subtract		add	add	subtract
81°	152	2,1827	9,4777	121°	3547	3,5499	9,7516
82	206	2,3139	9,4851	122	3675	3,5653	9,7602
83	261	2,4170	9,4924	123	3806	3,5805	9,7690
84	318	2,5021	9,4996	124	3941	3,5956	9,7781
85	376	2,5750	9,5067	125	4078	3,6105	9,7874
86	435	2,6386	9,5138	126	4220	3,6253	9,7971
87	496	2,6951	9,5207	127	4364	3,6399	9,8070
88	558	2,7470	9,5275	128	4513	3,6545	9,8172
89	622	2,7936	9,5342	129	4667	3,6690	9,8278
90	688	2,8373	9,5409	130	4825	3,6835	9,8387
91	754	2,8776	9,5475	131	4989	3,6980	9,8500
92	823	2,9152	9,5540	132	5158	3,7125	9,8618
93	892	2,9506	9,5605	133	5332	3,7269	9,8740
94	963	2,9837	9,5670	134	5512	3,7413	9,8866
95	1036	3,0154	9,5734	135	5698	3,7557	9,8997
96	1110	3,0455	9,5797	136	5890	3,7701	9,9132
97	1186	3,0742	9,5860	137	6088	3,7845	9,9273
98	1264	3,1016	9,5923	138	6297	3,7991	9,9419
99	1343	3,1279	9,5986	139	6513	3,8138	9,9571
100	1423	3,1531	9,6050	140	6739	3,8286	9,9729
101	1505	3,1774	9,6113	141	6974	3,8435	9,9893
102	1588	3,2008	9,6176	142	7219	3,8585	10,0064
103	1673	3,2236	9,6239	143	7475	3,8736	10,0242
104	1761	3,2457	9,6302	144	7743	3,8889	10,0427
105	1850	3,2671	9,6366	145	8024	3,9044	10,0620
106	1941	3,2879	9,6430	146	8316	3,9199	10,0822
107	2033	3,3081	9,6495	147	8630	3,9360	10,1032
108	2127	3,3278	9,6560	148	8962	3,9524	10,1251
109	2223	3,3469	9,6627	149	9312	3,9690	10,1481
110	2321	3,3656	9,6694	150	9681	3,9859	10,1720
111	2421	3,3840	9,6763	151	10072	4,0031	10,1970
112	2523	3,4019	9,6831	152	10491	4,0208	10,2231
113	2627	3,4195	9,6901	153	10940	4,0390	10,2504
114	2734	3,4368	9,6972	154	11416	4,0575	10,2792
115	2843	3,4538	9,7044	155	11929	4,0766	10,3094
116	2954	3,4704	9,7119	156	12483	4,0963	10,3412
117	3068	3,4868	9,7195	157	13086	4,1168	10,3746
118	3184	3,5030	9,7272	158	13740	4,1380	10,4098
119	3302	3,5188	9,7352	159	14455	4,1600	10,4469
120	3422	3,5345	9,7433	160	15223	4,1825	10,4861



AN ATTEMPT to shew the Advantage arising by
Taking the Mean of a Number of Observations,
in practical Astronomy.

HOUGH the method practised by *Astronomers*, in order to diminish the errors arising from the imperfection of instruments and of the organs of sense, by taking the mean of several observations, is of very great utility, and almost universally followed, yet has it not, that I know of, been hitherto subjected to any kind of demonstration.

In this Essay, some light is attempted to be thrown on the subject, from mathematical principles: in order to the application of which, it seemed necessary to lay down the following suppositions.

1. That there is nothing in the construction, or position of the instrument whereby the errors are constantly made to tend the same way, but that the respective chances for their happening in excess, and in defect, are either accurately, or nearly, the same.

2. That there are certain assignable limits between which all these errors may be supposed to fall; which limits depend on the goodness of the instrument and the skill of the observer.

These particulars being premised, I shall deliver what I have to offer on the subject, in the following Propositions.

PROPOSITION I.

Supposing that the several chances for the different errors that any single observation can admit of, are expressed by the terms of the series $r^{-\infty}$, r^{-3} , r^{-2} , r^{-1} , r^0 , r^1 , r^2 , r^3 , r^{∞} ; where the exponents denote the quantities and qualities of the respective errors, and the terms themselves, the respective chances for their happening; it is proposed to determine the probability, or odds,

odds, that the error, by taking the mean of a given number (n) of observations, exceeds not a given quantity $(\frac{m}{n})$.

It is evident from the laws of chance, that, if the given series $r^{-v} \dots + r^{-3} + r^{-2} + r^{-1} + r^0 + r^1 + r^2 + r^3 \dots + r^v$, expressing all the chances in one observation, be raised to the n th power, the terms of the series thence arising will truly exhibit all the different chances in all the proposed (n) observations. But in order to raise this power, with the greatest facility,

our given series may be reduced to $r^{-v} \times \frac{1 - r^{2v+1}}{1 - r}$ (by the

known rule for summing up the terms of a geometrical progression); whereof the n th power (making $w = 2v + 1$) will be $r^{-nv} \times \frac{1 - r^w}{1 - r} \times \frac{1 - r^w}{1 - r}$; which expanded, becomes

$$r^{-nv} - nr^{nw-nv} + \frac{n}{1} \cdot \frac{n-1}{2} r^{2w-nv} - \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{3w-nv} + \mathcal{E}c.$$

$$\times \text{ into } 1 + nr + \frac{n}{1} \cdot \frac{n+1}{2} r^2 + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} r^3 + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot$$

$$\frac{n+3}{4} r^4 + \mathcal{E}c.$$

Now, to find from hence the sum of all the chances, whereby the excess of the positive errors above the negative ones, can amount to a given number m precisely, it will be sufficient (instead of multiplying the former series by the whole of the latter) to multiply by such terms of the latter only, as are necessary to the production of the given exponent m , in question.

Thus the first term (r^{-nv}) of the former series, is to be multiplied by that term of the second whose exponent is $nv + m$, in order that the power of r , in the product, may be r^m : but it is plain, from the law of the series, that the coefficient of

this term (putting $nv + m = q$) will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q)$,

q being the number of factors; and, consequently, that the product

under consideration will be $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q) \times r^m$.

Again, the second term of the former series being $-nr^{nw-nv}$, the exponent of the corresponding term of the latter must therefore be $-w + nv + m (= q - w)$, and the term itself,

self, $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q - w) \times r^{n-w}$; which, drawn into $-nr^{n-w}$, gives $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q - w) \times nr^m$ for the second term required.

In like manner the third term of the product whose exponent is m , will be found, $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q - 2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m$. And the sum of all the terms, having the same, given exponent, will consequently be

$$\begin{aligned}
 & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q) \times r^m \\
 & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q - w) \times nr^m \\
 & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q - 2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m \\
 & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} \cdot \frac{n+3}{4} (q - 3w) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^m \\
 & \quad \text{\textit{\&c.}} \quad \quad \quad \text{\textit{\&c.}}
 \end{aligned}$$

From which general expression, by expounding m by 0, $+1$, -1 , $+2$, -2 &c. successively, the sum of the several chances whereby the difference of the positive and negative errors can fall within the proposed limits ($+m$, $-m$) will be found: which divided by the total of all the chances, or $r^{-nw} \times \overline{1 - r^w}^n \times \overline{1 - r}^{n-w}$, will be the true measure of the probability sought. From whence the advantage, by taking the mean of several observations, might be made to appear: but this will be shewn more properly in the next Proposition; which is better adapted, and to which this is premised, as a *Lemma*.

REMARK.

If r be taken $= 1$, or the chances for the positive, and the negative errors be supposed accurately the same; then our expression, by expunging the powers of r , will be the very same with that shewing the chances for throwing $n + q$ points, precisely, with n dice, each die having as many faces (w) as the result of any one single observation can come out different ways. Which may be made to appear, independent of any kind

kind of calculation, from the bare consideration, that the chances for throwing, precisely, the number m , with n dice, whereof the faces, of each, are numbered $-v \dots -3, -2, -1, -0, +1, +2, +3 \dots +v$, must be the very same as the chances whereby the positive errors can exceed the negative ones by that precise number: but the former are, evidently, the same as the chances for throwing precisely the number $v+1 \times n + m$ (or $n+q$) with the same n dice, when they are numbered in the common way, with the terms of the natural progression $1, 2, 3, 4, 5$, and so on; because the number upon each face being, *here*, increased by $v+1$, the whole increase upon all the (n) faces will be expressed by $v+1 \times n$; so that there will be, *now*, the very same chances for the number $v+1 \times n + m$, as there was *before* for the number m ; since the chances for throwing any faces assigned will continue the same, however those faces are numbered.

PROPOSITION II.

Supposing the respective chances for the different errors, which any single observation can admit of, to be expressed by the terms of the series $r^{-v} + 2r^{1-v} + 3r^{2-v} \dots + v+1 \cdot r^0 \dots 3r^{v-2} + 2r^{v-1} + r^v$ (whereof the coefficients, from the middle one ($v+1$) decrease both ways, according to the terms of an arithmetical progression); it is proposed to find the probability, or odds, that the error, by taking the mean of a given number (t) of observations, exceeds not a given quantity $\left(\frac{m}{t}\right)$.

Following the method laid down in the preceding proposition, the sum, or value of the series here proposed will appear

to be $\frac{r^{-v} \times 1 - r^{v+1}}{1-r}$ (being the same with the square of the

geometrical progression $r^{-\frac{1}{2}} \times 1 + r + r^2 + r^3 \dots + r^v$). And the power thereof whose exponent is t (by making $n = 2t$, and $w = v+1$) will therefore be $r^{-tv} \times 1 - r^{n \times t} \times 1 - r^{-n}$
 $= r^{-tv} - nr^{w-tv} + \frac{n}{1} \cdot \frac{n-1}{2} r^{2w-tv} - \&c.$ into $1 + nr + \frac{n}{1} \cdot$

$\frac{n+1}{2}r^2 + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3}r^3 + \mathcal{E}c.$ Which two series's being the same with those in the preceding Problem (excepting only, that the exponents in the former of them are expressed in terms of t , instead of n) it is plain, that, if q be here put $= tv + m$ (instead of $nv + m$) the conclusion there brought out will answer equally here; and consequently that the sum of all the chances, whereby the excess of the positive errors, above the negative ones, can amount to the given number m , precisely, will here, also, be truly defined by

$$\begin{aligned} & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q) \times r^m \\ & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w) \times nr^m \\ & + \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-2w) \times \frac{n}{1} \cdot \frac{n-1}{2} r^{2m} \\ & - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-3w) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^{3m} \\ & \mathcal{E}c. \qquad \mathcal{E}c. \end{aligned}$$

But this general expression, as several of the factors in the numerators and denominators mutually destroy each other, may be transformed to another more commodious.

Thus the quantity $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q)$, in the first line, by breaking the numerator and denominator, each into two parts, will become

$$\frac{n \cdot n+1 \cdot n+2 \cdot n+3 \cdot \dots \cdot q \cdot q+1 \cdot q+2 \cdot q+3 \cdot \dots \cdot q+n-1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot n \cdot n+1 \cdot n+2 \cdot n+3 \cdot \dots \cdot q}$$

which, by equal division, is reduced to

$$\frac{q+1 \cdot q+2 \cdot q+3 \cdot \dots \cdot q+n-1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n-1} = \frac{q+n-1 \cdot q+n-2 \cdot q+n-3 \cdot \dots \cdot q+1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n-1} = \frac{p-1 \cdot p-2 \cdot p-3 \cdot \dots \cdot (n-1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1)}$$

supposing $p = q + n = tv + m + n$.

In the very same manner, by making $q' = q - w$, and $p' = q' + n (= p - w)$ it appears that $\frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+2}{3} (q-w)$

$= \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n-1) \mathcal{E}c.$ Consequently our whole given expression (making $p'' = p - 2w$, $p''' = p - 3w$, $\mathcal{E}c.$) will be transformed to

$$\begin{aligned} & + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} (n-1) \times r^m \\ & - \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n-1) \times nr^m \\ & + \frac{p''-1}{1} \cdot \frac{p''-2}{2} \cdot \frac{p''-3}{3} (n-1) \times \frac{n}{1} \cdot \frac{n-1}{2} r^m \\ & - \frac{p'''-1}{1} \cdot \frac{p'''-2}{2} \cdot \frac{p'''-3}{3} (n-1) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} r^m \\ & \mathcal{E}c. \qquad \mathcal{E}c. \end{aligned}$$

Which expression is to be continued till some of the factors become nothing, or negative; and which, when $r=1$, is the very same with that exhibiting the number of chances for p points, precisely, on n dice, having each w faces.

And, in this case, where the chances for the errors in excess and in defect are the same, the solution is the most simple it can be; since, from the chances above determined, answering to the number p precisely, the sum of the chances for all the inferior numbers to p , may be readily obtained, being given

(from the method of increments) equal to $\frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} (n)$

$$\begin{aligned} & - \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n) \times n + \frac{p''-1}{1} \cdot \frac{p''-2}{2} \cdot \frac{p''-3}{3} (n) \times \frac{n}{1} \cdot \frac{n-1}{2} \\ & - \frac{p'''-1}{1} \cdot \frac{p'''-2}{2} \cdot \frac{p'''-3}{3} (n) \times \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + \mathcal{E}c. \end{aligned}$$

The difference between which and half (w^n) the sum of all the chances (which difference I shall denote by D) will consequently be the true number of the chances whereby the errors in excess (or in defect) can fall within the given limit (m): so

that $\frac{D}{\frac{1}{2}w^n}$ will be the true measure of the required probability, that the error, by taking the mean of t observations, exceeds not the quantity $\frac{m}{t}$ proposed.

But

Of the Advantage arising by Taking the Mean

But now, to illustrate this by an example, from whence the utility of the method in practice may clearly appear, it will be necessary, in the first place, to assign some number for v , expressing the limits of the errors to which any observation is subject. These limits indeed (as has been before observed) depend on the goodness of the instrument, and the skill of the observer: but I shall here suppose, that every observation may be relied on, to five seconds; and that the chances for the several errors $-5''$, $-4''$, $-3''$, $-2''$, $-1''$, $0''$, $+1''$, $+2''$, $+3''$, $+4''$, $+5''$, included within the limits thus assigned, are respectively proportional to the terms of the series 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1. Which series is much better adapted, than if all the terms were to be equal; since it is highly reasonable to suppose, that the chances for the respective errors decrease, in proportion as the errors themselves increase.

These particulars being premised, let it be now required to find what the probability, or chance for an error of 1, 2, 3, 4, or 5 seconds will be, when, instead of relying on one, the mean of six observations is taken.

Here, v being $= 5$, and $t = 6$, we shall have $n (= 2t) = 12$, $w (= v + 1) = 6$, and $p (= tv + n + m) = 42 + m$: but the value of m , if we first seek the chances whereby the error exceeds not one second, will be had from the equation $\frac{m}{t} = \pm 1$; where either sign may be used (the chances being the same) but the negative one is the most commodious: from whence we have $m (= -t) = -6$; and therefore $p = 36$, $p' = 30$, $p'' = 24$, &c.

Which values being substituted in the general expression above determined, it will become $\frac{35}{1} \cdot \frac{34}{2} \cdot \frac{33}{3} (12) - \frac{29}{1} \cdot \frac{28}{2} \cdot \frac{27}{3} (12) \times 12 + \frac{23}{1} \cdot \frac{22}{2} \cdot \frac{21}{3} (12) \times 66 - \frac{17}{1} \cdot \frac{16}{2} \cdot \frac{15}{3} (12) \times 220 = 299576368$: and this subtracted from $1088391168 (= \frac{1}{6} \times 6^{12})$, leaves 78881480 , for the value of D corresponding: therefore the required probability that the error, by taking the mean of six observations, exceeds not a single *second*, will be truly measured by the fraction $\frac{78881480}{1088391168}$; and consequently the odds will

will be as 788814800 to 299576368, or nearly as $2\frac{1}{2}$ to 1. But the odds, or proportion, when one single observation is taken, is only as 16 to 20, or as $\frac{8}{10}$ to 1.

To determine, now, the probability that the result comes within two seconds of the truth, let $\frac{m}{t}$ be made $= -2$; so shall $m (= -2t) = -12$: therefore $p = 30$, $p' = 24$, $p'' = 18$, &c. and our general expression will here come out $= 36079407$; whence $D = 1052311761$. Consequently $\frac{1052311761}{1088391168}$ will be the true measure of the probability sought: and the odds, or proportion of the chances, will therefore be that of 1052311761 to 36079407, or as 29 to 1, nearly. But the proportion, or odds, when a single observation is taken, is only as 2 to 1: so that the chance for an error exceeding two seconds, is not $\frac{1}{10}$ th part so great, from the mean of six, as from one single observation. And it will be found in the same manner, that the chance for an error exceeding three seconds is not *here* $\frac{1}{1000}$ part so great as it will be from one observation only. Upon the whole of which it appears, that, the taking of the mean of a number of observations, greatly diminishes the chances for all the smaller errors, and cuts off almost all possibility of any large ones: which last consideration alone is sufficient to recommend the use of the method, not only to *Astronomers*, but to all *Others* concerned in making experiments, or observations of any kind, which will allow of being repeated under the same circumstances.

In the preceding calculations, the different errors to which any observation is supposed subject, are restrained to whole quantities, or a certain, precise, number of seconds; it being impossible, from the most exact instruments, to take off the quantity of an angle to a *geometrical exactness*. But I shall now shew how the chances may be computed, when the error admits of any value whatever, whole or broken, within the proposed limits, or when the result of each observation is supposed to be *accurately* known.

Let

Fig. 20. Let, then, the line AB represent the whole extent of the given interval, within which all the observations are supposed to fall; and conceive the same to be divided into an exceeding great number of very small, equal particles, by perpendiculars terminating in the sides AD, BD of an isosceles triangle ABD, formed upon the base AB: and let the probability or chance whereby the result of any observation tends to fall within any of these very small intervals Nn , be proportional to the corresponding area $NMnm$, or to the perpendicular NM ; then, since these chances (or areas) reckoning from the extremes A and B, increase according to the terms of the arithmetical progression 1, 2, 3, 4, &c. it is evident that the case is here the same with that in the latter part of Prop. II.; only, as the number v (expressing the particles in AC or BC) is indefinitely great, all (finite) quantities joined to v , or its multiples, with the signs of addition or subtraction, will here vanish, as being nothing in comparison of v . By which means the general expression $\left(\frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3} (n) - \frac{p'-1}{1} \cdot \frac{p'-2}{2} \cdot \frac{p'-3}{3} (n) \times n + \frac{p''-1}{1} \cdot \frac{p''-2}{2} \cdot \frac{p''-3}{3} (n) \times n \cdot \frac{n-1}{2}, \&c.\right)$ there determined, will here become $\frac{p}{1} \cdot \frac{p}{2} \cdot \frac{p}{3} (n) - \frac{p'}{1} \cdot \frac{p'}{2} \cdot \frac{p'}{3} (n) \times n, \&c. = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 (n)}$
 $\times p^n - np^n + n \cdot \frac{n-1}{2} p''^n - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} p'''^n, \&c.$ (wherein $p = tv + m$, $p' = p - v$, $p'' = p - 2v$, $p''' = p - 3v$, &c.) and therefore, the value of D in the present case, being $\frac{1}{2}v^n - \frac{1}{1 \cdot 2 \cdot 3 (n)} \times p^n - n \cdot \overline{p - v}^n + n \cdot \frac{n-1}{2} \cdot \overline{p - 2v}^n, \&c.$ it is evident that the probability $\left(\frac{D}{\frac{1}{2}v^n}\right)$ of the error's not exceeding the quantity $\frac{m}{t}$ (in taking the mean of t observations) will be truly defined by

$$1 - \frac{2}{1 \cdot 2 \cdot 3 (n)} \times \left[\frac{p}{v} \right]^n - n \times \left[\frac{p}{v} - 1 \right]^n + \frac{n \cdot n-1}{1 \cdot 2} \times \left[\frac{p}{v} - 2 \right]^n, \&c.$$

which may be represented by the curvilineal area CNFE, corresponding

responding to the given value or abscissa CN ($= \frac{m}{t}$). Now, though the numbers v , p , and m are, all of them, here supposed to be indefinitely great, yet they may be exterminated, and the value of the expression determined, from their known relation to each other. For if the given ratio of $\frac{m}{t}$ to v , or of CN to CA, be expressed by that of x to 1, or, which is the same, if the error in question be supposed the x part of the greatest error; then, m being $= tvx$, p ($= tv \mp m$) will be $= tv \mp tvx$, and therefore $\frac{p}{v} = t \times 1 \mp x$; which let be denoted by y : then, by substitution, our last general expression will become

$$1 - \frac{2}{1 \cdot 2 \cdot 3 (n)} \times \left\{ y^n - n \cdot y^{n-1} + \frac{n \cdot n-1}{1 \cdot 2} \cdot y^{n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \cdot y^{n-3} \right\}, \text{ \&c.}$$

which series is to be continued till the quantities y , $y-1$, $y-2$, &c. become negative.

As an example of what is above delivered, let it be now required to find the probability, or odds that the error, by taking the mean of six observations, exceeds not a single second; supposing (as in the former example) that the greatest error, that any observation can admit of, is limited to five seconds.

Here t being $= 6$, n ($= 2t$) $= 12$, and $x = \frac{1}{5}$, we have y ($= t \times 1 - x$) $= 4,8$; and therefore the measure of the probability sought will be equal to $1 - \frac{2}{1 \cdot 2 \cdot 3 (12)} \times$

$4,8^{12} - 12 \times 3,8^{12} + 66 \times 2,8^{12} - 220 \times 1,8^{12} + 495 \times 0,8^{12}$
 $= 0,7668$, nearly: so that the odds, that the error exceeds not a single second, will be as 0,7668 to 0,2332; which is more than three to one. But the proportion, when one single observation is relied on, is only as 36 to 64, or as 9 to 16.

In the same manner, taking $x = \frac{2}{5}$, it will be found, that

the odds, of the error's not exceeding two seconds, when the mean of six observations is taken, will be as 0,985 to 0,015, nearly, or as $65\frac{2}{3}$ to 1; whereas the odds on one single observation, is only as 64 to 36, or as $1\frac{7}{9}$ to 1: so that the chance for an error of two seconds is not $\frac{1}{20}$ th part so great, from the mean of six, as from one single observation. And it will farther appear, by making $x = \frac{3}{5}$, that the probability of an error of three seconds, here, is not $\frac{1}{1400}$ th part so great as from one single observation: so that in this, as well as in the former hypothesis, almost all possibility of any large error is cut off. And the case will be found the same, whatever hypothesis is assumed to express the chances for the errors to which any single observation is subject.

From the same general expression by which the foregoing proportions are derived, it will be easy to determine the odds, that the mean of a given number of observations is nearer to the truth than one single observation, taken indifferently. For, if z be put $(= 1 - x) = \frac{y}{t}$, and $s = \frac{1}{t}$, then, y being $= tz$, the quantity

$$\frac{2^n}{1.2.3(n)} \times y^n - n.y - 1^n + n.\frac{n-1}{2}.y - 2^n, \&c. \text{ (expressing the probability that the result falls within the distance } z \text{ of the greatest limit) will here, by substitution, become}$$

$$\frac{2t^n}{1.2.3(n)} \times z^n - n \times z - 1^n + n.\frac{n-1}{2} \times z - 2^n, \&c. \text{ which, in case of one single observation (when } t = 1, \text{ and } n = 2) \text{ is barely } z^2, \text{ and its fluxion } 2zz: \text{ therefore, if we now multiply by } 2zz, \text{ the product}$$

$$\frac{4t^n}{1.2.3(n)} \times z^{n+1}z - n.z - 1^n.zz + n.\frac{n-1}{2}.z - 2^n.zz, \&c. \text{ will give the fluxion of the probability that the result of } t \text{ observations is farther from the truth, or nearer to the limits, than one single observation taken indifferently. And consequently the fluent thereof, which is } \frac{4t^n}{1.2.3(n)} \text{ into } \frac{z^{n+2}}{n+2}$$

$$-\frac{n}{1} \times \frac{\sqrt{s \cdot z - s}^{n+1}}{n+1} + \frac{\sqrt{z - s}^{n+2}}{n+2} + \frac{n}{1} \cdot \frac{n-1}{2} \times \frac{\sqrt{2s \cdot z - 2s}^{n+1}}{n+1} + \frac{\sqrt{z - 2s}^{n+2}}{n+2}$$

Ec. will, when $z = 1$, be the true measure of the probability itself. Which, in the case above proposed, where $t = 6$, and $n = 12$, will be found $= 0,245$, and, consequently, the odds that the mean of six, is nearer to the truth than one single observation, as 755 to 245, or as 151 to 49.





A

DETERMINATION

OF

Certain FLUENTS, and the RESOLUTION of some very useful EQUATIONS in the higher Orders of Fluxions ; by means of the Measures of Angles and Ratios, and the Right-fines and Verfed-fines of circular Arcs.

IN order to treat the matter here propofed with due
 * I * perfpicuity, it will be neceffary, previous thereto, to
 * * * give a demonftration of the two fubfequent Lemmas.

LEMMA I.

The double of the rectangle contained under the co-fines of any two arcs, fuppoſing the radius to be unity, is equal to the ſum of the co-fines of the ſum, and difference of thoſe arcs.

Fig. 21. For, let AB and BD (= BE) be the two arcs, and CG and Cn their reſpective co-fines ; likewise let CH be the co-fine of their ſum AE, and CF that of their difference AD ; making nm parallel to BG. Then, Dn being = En, it follows that Fm = Hm ; and conſequently that $2Cm = CH + CF$: but, by ſimilar triangles, $CB : CG :: Cn : Cm$; whence $CG \times 2Cn = CB \times 2Cm = CB \times \overline{CH + CF}$. Q. E. D.

LEMMA II.

If A be any arch of a circle whoſe radius is unity, and n any whole poſitive number ; then will

$$\overline{\text{co-fin. } A}^n = \left[\frac{1}{2} \right]^{n-1} \text{ into co-fin. } nA + n \text{ co-fin. } \overline{n-2} \cdot A + n \cdot \frac{n-1}{2} \text{ co-fin. } \overline{n-4} \cdot A + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \text{ co-fin. } \overline{n-6} \cdot A + \text{Ec. continued}$$

to $\frac{n+1}{2}$, or $\frac{1}{2}n+1$ terms, according as n is an odd, or even number; in the latter of which cases the half, only, of the last term is to be taken.

For, by the preceding Lemma, $2 \overline{\text{cof. A}}^2 = \text{cof. A} + \text{A} + \text{cof. A} - \text{A} = \text{cof. 2A} + 1$; which equation multiplied by 2cof. A , gives $2^2 \times \overline{\text{cof. A}}^3 = \text{cof. 2A} \times 2 \text{cof. A} + 2 \text{cof. A} = \text{cof. 3A} + \frac{\text{cof. A}}{2 \text{cof. A}}$ (by Lem. I.) $= \text{cof. 3A} + 3 \text{cof. A}$.

Multiply, again, by 2cof. A ; so shall $2^3 \times \overline{\text{cof. A}}^4 = \text{cof. 3A} \times 2 \text{cof. A} + 3 \text{cof. A} \times 2 \text{cof. A} = \text{cof. 4A} + \frac{\text{cof. 2A}}{\text{cof. 2A}} + 3 \text{cof. 0}$ (by Lemma I.) $= \text{cof. 4A} + 4 \text{cof. 2A} + 3$. In the same manner we have,

$$\begin{aligned} 2^4 \times \overline{\text{cof. A}}^5 &= \text{cof. 5A} + 5 \text{cof. 3A} + 10 \text{cof. A}, \\ 2^5 \times \overline{\text{cof. A}}^6 &= \text{cof. 6A} + 6 \text{cof. 4A} + 15 \text{cof. 2A} + 10, \\ 2^6 \times \overline{\text{cof. A}}^7 &= \text{cof. 7A} + 7 \text{cof. 5A} + 21 \text{cof. 3A} + 35 \text{cof. A}, \\ &\quad \&c. \qquad \qquad \qquad \&c. \end{aligned}$$

where the law of continuation is manifest; the numeral coefficients being the same, and generated in the very same manner, with those of a binomial raised to the 2d, 3d, 4th, 5th, &c. powers, successively; except in the last term, when the exponent n is even, in which case one half only of the correspondent (or middle) term of the involved binomial is concerned. Hence the proposition is manifest.

The same otherwise.

If the co-sine of A be denoted by x , it is well known that $\dot{A} = \frac{-\dot{x}}{\sqrt{1-xx}}$. Multiply the whole equation by $\sqrt{-1}$, so

$$\text{shall } \dot{A}\sqrt{-1} = \frac{-\dot{x}\sqrt{-1}}{\sqrt{1-xx}} = \frac{-\dot{x}x - 1}{\sqrt{1-xx} \times \sqrt{-1}} = \frac{\dot{x}}{\sqrt{-1+xx}};$$

whence, by taking the fluent, we have $A\sqrt{-1} = \text{hyp. log. } x + \sqrt{xx} - 1$. Let N be the number whose hyperbolical logarithm

arithm is 1; then, since $\text{hyp. log. } N^{\sqrt{-1}} (= A\sqrt{-1}) = \text{hyp. log. } x + \sqrt{xx - 1}$, it is evident that $N^{\sqrt{-1}} = x + \sqrt{xx - 1}$, or $M^A = x + \sqrt{xx - 1}$ (by making $M = N^{\sqrt{-1}}$). From which equation x (the co-sine of A) is found $= \frac{M^A + M^{-A}}{2}$ (and from thence $(\sqrt{1 - xx})$ the sine of $A = \sqrt{-1} \times \frac{M^A - M^{-A}}{2}$; but this last by the bye).

Now seeing that $2 \text{ cof. } A = M^A + M^{-A}$, we therefore have $2 \text{ cof. } A]^n = \overline{M^A + M^{-A}}^n = \overline{M^{nA} + M^{-nA}} + n \times \overline{M^{n-2.A} + M^{-n+2.A}} + n \cdot \frac{n-1}{2} \times \overline{M^{n-4.A} + M^{-n+4.A}} + \mathcal{E}c.$ by expanding $\overline{M^A + M^{-A}}^n$ and uniting, in pairs, the correspondent terms (*viz.* the first and last, the second and last but one, and so on). But $M^{nA} + M^{-nA}$, the first of these pairs, is the double of the co-sine of nA ; for the very same reason that $M^A + M^{-A}$ was found to express the double of the co-sine of A . And thus, $\overline{M^{n-2.A} + M^{-n+2.A}}$ will appear to express the double of the co-sine of $n-2.A$, &c. And our equation will therefore be reduced to $2^n \times \text{cof. } A]^n = 2 \text{ cof. } nA + 2n \text{ cof. } n-2.A + 2n \cdot \frac{n-1}{2} \text{ cof. } n-4.A + \mathcal{E}c.$ or to $\text{cof. } A]^n = \left[\frac{1}{2}\right]^{n-1}$

$$\times \left\{ \begin{aligned} &\text{cof. } nA + n \text{ cof. } n-2.A + n \cdot \frac{n-1}{2} \text{ cof. } n-4.A \\ &+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \text{ cof. } n-6.A + \mathcal{E}c. \end{aligned} \right.$$

where, when the exponent n is even (the number of terms in $\overline{M^A + M^{-A}}^n$, expanded, being odd) there will be a middle term (no-ways effected by M or A) which being an absolute number, must be taken singly, and consequently, only the half thereof when the whole series is divided by 2, as is the case in the conclusion. *Q. E. D.*

COROLLARY.

If Q be taken to represent an arch of 90 degrees, and the complement ($Q - A$) of the arch A be put $= B$; then, by substituting $\sin. B$ for $\cos. A$, and $Q - B$ for A in our general equation, we shall have $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1}$ into $\cos. nQ - nB$

$$+ n \cos. n-2 . Q - n-2 . B + n . \frac{n-1}{2} \cos. n-4 . Q - n-4 . B$$

+ &c. being a general expression for any power of the *sine* of an arch (as the former was of the *co-sine*). But this expression may be reduced to a form somewhat more commodious, regard being had to the different interpretations of n , with respect to even, and odd numbers. Thus, if n be expounded by any term of the series 4, 8, 12, 16, &c. it is evident that nQ (in the first term) will be an *even* multiple of the semi-periphery; and that $n-2 . Q$ (in the second term) will be an *odd* one, and so on, alternately. But it is well known, that subtracting, or casting off any multiple of the semi-periphery no-ways affects the value of the sine, or co-sine; except, that such value, when the multiple is an odd one, will be changed from positive to negative (and *vice versa*). Hence our last equation will be reduced to $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1}$ into $\cos. - nB -$

$$n \cos. - n-2 . B + n . \frac{n-1}{2} \cos. n-4 . B - \&c. =$$

$$\left(\frac{1}{2}\right)^{n-1} \text{ into } \cos. nB - n \cos. n-2 . B + n . \frac{n-1}{2} \cos. n-4 . B$$

&c.

And, for the same reasons, the equation, when n is interpreted by any term of the series 2, 6, 10, 14, &c. will appear to be $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1}$ into $-\cos. nB + n \cos. n-2 . B$

$$- n . \frac{n-1}{2} \cos. n-4 . B + \&c.$$

But, when n is expounded by any of the odd numbers 1, 5, 9, 13, &c. we shall then (by rejecting the multiples of the
semi-

semi-periphery, &c.) have $\overline{\sin.B}^n = \frac{1}{2} \left| \right|^{n-1}$ into $\text{cof. } \overline{Q-nB}$
 $- n \text{ cof. } \overline{Q-n-2.B} + n \cdot \frac{n-1}{2} \text{ cof. } \overline{Q-n-4.B} -$
 $\&c. = \frac{1}{2} \left| \right|^{n-1}$ into $\text{fin. } nB - n \text{ fin. } \overline{n-2.B} + n \cdot \frac{n-1}{2} \text{ fin. } \overline{n-4.B} - \&c.$

Lastly, if n be expounded by any term of the series 3, 7, 11, 15, &c. the result, or series, will be the same as in the preceding case, only the signs of all the terms must be changed to their contrary.

But all these different cases may be otherwise, more directly, investigated, by means of the two following Theorems; whereof the Demonstration is obvious, from that of *Lemma I**.

1°. The double of the rectangle contained under the sines of any two arcs, supposing the radius to be unity, is equal to the difference of the co-sines of the sum, and difference of those arcs.

2°. And the double of the rectangle under the co-sine of the one and the sine of the other, is equal to the difference of the sines of the sum, and difference of the said arcs.

Hence it follows, that $\sin.B \times 2 \sin.B = -\text{cof. } \overline{B+B} + \text{cof. } \overline{B-B}$ (by *Theor. I.*) $= -\text{cof. } 2B + 1$: whence, multiplying the whole equation by $2 \sin.B$, we have $2^2 \times \overline{\sin.B}^3 = -\text{cof. } 2B \times 2 \sin.B + 2 \sin.B = -\text{fin. } 3B + \frac{\text{fin. } B}{2 \sin.B}$ (by *Theor. II.*) $= -\text{fin. } 3B + 3 \sin.B$. Whence, again, by equal multiplication, $2^3 \times \overline{\sin.B}^4 = -\text{fin. } 3B \times 2 \sin.B + 3 \sin.B \times 2 \sin.B = +\text{cof. } 4B - \frac{\text{cof. } 2B}{3} + 3 \text{ cof. } 0$ (by *Theorem I.*) $= \text{cof. } 4B - 4 \text{ cof. } 2B + 3$.

In like manner, $2^4 \times \overline{\sin.B}^5 = \text{fin. } 5B - 5 \text{ fin. } 3B + 10 \text{ fin. } B$; and $2^5 \times \overline{\sin.B}^6 = -\text{cof. } 6B + 6 \text{ cof. } 4B - 15 \text{ cof. } 2B + 10$, &c.

Fig. 21.

* By $\sin.\Delta$'s, $BC(1) : BG :: DE(2Dn) : Dp(CF-CH) = 2Dn \times BG$:
 And $BC(1) : CG :: DE(2Dn) : Ep(EH-DF) = 2CG \times Dn$.
 Whence,

Whence, universally, $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1} \times$

$$\pm \sin. nB \mp n \sin. \overline{n-2} . B \pm n \cdot \frac{n-1}{2} \sin. \overline{n-4} . B \mp \&c.$$

when n is an odd number; and $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1} \times$

$$\pm \cos. nB \mp n \cos. \overline{n-2} . B \pm n \cdot \frac{n-1}{2} \cos. \overline{n-4} . B \mp \&c.$$

when n is an even number: where the number of terms, in the former case, will be $\frac{n+1}{2}$, and in the latter $\frac{1}{2}n + 1$; in which case the half, only, of the last term is to be taken; and is always positive, as well as the last term in the former case: whereby the signs of all the other terms (as they change alternately) will be known.

If $\alpha, \beta, \gamma, \delta \dots M$ be assumed to express the terms ($1, n, n \cdot \frac{n-1}{2}, n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}, \&c.$) of $1 + 1$ raised to the n th power (M being the middle, or greatest term) it is evident that the second case of our general equation (wherein n is even) will stand thus, $\overline{\sin. B}^n = \left(\frac{1}{2}\right)^{n-1}$ into $\pm \alpha \cos. nA \mp \beta \cos. \overline{n-2} . A \pm \gamma \cos. \overline{n-4} . A \mp \delta \cos. \overline{n-6} . A \dots + \frac{1}{2}M$.

By the same method of proceeding, an expression exhibiting the continual product of the co-fines, or sines of any number of unequal arches, may be derived.

For (by Lemma I.) $\cos. A \times 2 \cos. B = \cos. \overline{A+B} + \cos. A - B$; whence $\cos. A \times 2 \cos. B \times 2 \cos. C = \cos. \overline{A+B} \times 2 \cos. C + \cos. \overline{A-B} \times 2 \cos. C$ (by equal multiplication)

$$\begin{aligned} & \cos. \overline{A+B-C} \\ = & \cos. \overline{A+B+C} + \cos. \overline{A-B+C} \text{ (by the Lemma); } \\ & \cos. \overline{-A+B+C} \end{aligned}$$

whence, again (by equal multiplication and the Lemma) we have

M

have

The Resolution of certain fluxionary Equations;

have $\text{cof. } A \times 2 \text{ cof. } B \times 2 \text{ cof. } C \times 2 \text{ cof. } D = \text{cof.}$

$$\frac{\text{cof. } A+B+C-D}{A+B+C+D} + \frac{\text{cof. } A+B-C+D}{\text{cof. } A-B+C+D} + \frac{\text{cof. } A+B-C-D}{\text{cof. } -A+B+C-D}$$

from which the law of continuation is manifest.

PROBLEM I.

To determine the fluent of $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}}$; n being any odd affirmative number.

If A be assumed to denote the arch of a circle, whose sine is x and radius 1, it is well known that $\dot{A} = \frac{x}{\sqrt{1-xx}}$: and,

by the Corol. to Lem. II, it also appears that $x^n = \pm \left[\frac{1}{2} \right]^{n-1}$

$$\times \sin. nA - n \sin. \overline{n-2} . A + n . \frac{n-1}{2} \sin. \overline{n-4} . A \left(\frac{n+1}{2} \right).$$

Hence we have $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} (= x^n \dot{A}) = \pm \left[\frac{1}{2} \right]^n \times$

$$\dot{A} \times \sin. nA - n \dot{A} \times \sin. \overline{n-2} . A + n . \frac{n-1}{2} \dot{A} \times \sin. \overline{n-4} . A \left(\frac{n+1}{2} \right).$$

But the fluxion of any arch, multiplied by the sine (the radius being unity) is equal to the fluxion of the versed-sine: therefore the fluxions of the versed-sines of the arches nA , $\overline{n-2} . A$, $\overline{n-4} . A$, &c. will be $n \dot{A} \times \sin. nA$, $\overline{n-2} . \dot{A} \times \sin. \overline{n-2} . A$, $\overline{n-4} . \dot{A} \times \sin. \overline{n-4} . A$, &c. respectively; and consequently the said versed-sines, the true fluents of these fluxions: whence it is manifest that the true fluent of our

whole expression will be $\pm \left[\frac{1}{2} \right]^n$ into $\frac{1}{n} \times \text{vers. fin. } nA - \frac{n}{n-2}$
 $\times \text{vers. fin. } \overline{n-2} . A + \frac{n}{n-4} . \frac{n-1}{2} \times \text{vers. fin. } \overline{n-4} . A -$
 $\frac{n}{n-6} . \frac{n-1}{2} . \frac{n-2}{3} \times \text{vers. fin. } \overline{n-6} . A \left(\frac{n+1}{2} \right)$. Wherein, of
the

the signs $+$ and $-$ before $\left(\frac{1}{2}\right)^{n-1}$, the former, or latter obtains, according as $\frac{n+1}{2}$, expressing the number of terms, is odd or even.

PROBLEM II.

To find the fluent of $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}}$; n being any even affirmative number.

By the preceding Problem $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} = \dot{A}$; and, by the Corol.

to Lem. II, $x^n = \pm \left(\frac{1}{2}\right)^{n-1} x$

$$\alpha \text{ cof. } nA - \beta \times \text{cof. } \overline{n-2} . A + \gamma \times \text{cof. } \overline{n-4} . A \dots \pm \frac{1}{2} M.$$

Therefore $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} (= x^n \dot{A}) = \pm \left(\frac{1}{2}\right)^{n-1} x$

$$\alpha \dot{A} \times \text{cof. } nA - \beta \dot{A} \times \text{cof. } \overline{n-2} . A + \gamma \dot{A} \times \text{cof. } \overline{n-4} . A \dots \pm \frac{1}{2} M \dot{A}.$$

But the fluxion of any arch, multiplied by the co-sine, is equal to the fluxion of the sine, drawn into the radius: whence it follows that $n \dot{A} \times \text{cof. } nA$, $\overline{n-2} . \dot{A} \times \text{cof. } \overline{n-2} . A$, $\overline{n-4} . \dot{A} \times \text{cof. } \overline{n-4} . A$, &c. are the fluxions of the sines of the arches nA , $\overline{n-2} . A$, $\overline{n-4} . A$, &c. respectively; and, consequently that

$$\pm \left(\frac{1}{2}\right)^{n-1} x \left\{ \frac{\alpha}{n} \times \text{fin. } nA - \frac{\beta}{n-2} \times \text{fin. } \overline{n-2} . A + \frac{\gamma}{n-4} \times \text{fin. } \overline{n-4} . A - \frac{\delta}{n-6} \times \text{fin. } \overline{n-6} . A \dots \pm \frac{1}{2} M A \right.$$

will be the true fluent sought: where $\alpha = 1$, $\beta = n$, $\gamma = \beta \times \frac{n-1}{2}$, $\delta = \gamma \times \frac{n-2}{3}$, &c. and wherein the sign $+$ or $-$, before $\left(\frac{1}{2}\right)^{n-1}$, obtains according as $\frac{1}{2}n + 1$, expressing the number of terms, is odd, or even.

COROLLARY I.

Since the value (M) of the middle term of the binomial $(1+1)^n$, expanded in a series, is known to be $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \dots \frac{\frac{1}{2}n+1}{\frac{1}{2}n}$, by the law of the series, it is evident that the term next adjacent to it (on either side) will be expressed by $M \times \frac{\frac{1}{2}n}{\frac{1}{2}n+1}$, or by $M \times \frac{m}{m+1}$; making $m = \frac{1}{2}n$. And, in the same manner, it will appear that the next term to this last will be expressed by $M \times \frac{m}{m+1} \times \frac{m-1}{m+2}$; and the next to that, by

$M \times \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{m-2}{m+3}$; and so on. Therefore, by substituting these values above, and inverting the order of the terms, the general fluent, there given, will here be transformed to

$$\frac{M}{2^n} \times \left\{ A - \frac{m}{m+1} \times \sin. 2A + \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{1}{2} \sin. 4A - \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{m-2}{m+3} \times \frac{1}{3} f. 6A + \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{m-2}{m+3} \times \frac{m-3}{m+4} \times \frac{1}{4} f. 8A \right.$$

&c. where the series is to be continued till it terminates; and where the value of the general multiplier $\frac{M}{2^n}$ will be truly (and

most commodiously) expressed by $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \dots \times \frac{n-1}{n}$. For

$$M \text{ being } = \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \dots \frac{1}{2}n+1}{1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n}, \text{ if the numerator}$$

hereof be multiplied by $\frac{1}{2}n - 1 \cdot \frac{1}{2}n - 2 \cdot \frac{1}{2}n - 3 \dots 3 \cdot 2 \cdot 1$, and the denominator, at the same time, by its equal $1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n - 2 \cdot \frac{1}{2}n - 1 \cdot \frac{1}{2}n$, we shall then have $M =$

$$\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \dots 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n \times 1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \dots n}{1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n \times 1 \cdot 2 \cdot 3 \cdot 4 \dots \frac{1}{2}n};$$

$$\text{which divided by } 2^n \text{ gives } \frac{M}{2^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \dots n}{2 \cdot 4 \cdot 6 \cdot 8 \dots n \times 2 \cdot 4 \cdot 6 \cdot 8 \dots n}$$

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n} \cdot n$$

COROL-

COROLLARY II.

Hence may the fluent of $x^{\frac{n}{2}}\sqrt{1-xx}$ be likewise deduced; for this expression may be changed to $\frac{x^{\frac{n}{2}} \times \sqrt{1-xx}}{\sqrt{1-xx}}$, or to

$$\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} - \frac{x^{\frac{n}{2}+2}}{\sqrt{1-xx}}; \text{ whereof the fluent of the first term is}$$

$$\text{already found} = \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \dots \frac{n-1}{n} \times$$

$$A - \frac{m}{m+1} \times \text{fin. } 2A + \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{1}{2} \text{fin. } 4A, \text{ \&c. And, by making } n' = n+2, \text{ and } m' = \frac{1}{2}n' (=m+1), \text{ the fluent of the second term} - \frac{x^{\frac{n}{2}+2}}{\sqrt{1-xx}} \left(= - \frac{x^{\frac{n'}{2}}}{\sqrt{1-xx}} \right) \text{ in the same}$$

$$\text{manner is given equal to} - \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \dots \frac{n'-1}{n'} \times$$

$$A - \frac{m'}{m'+1} \times \text{fin. } 2A, \text{ \&c.} = - \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \dots \frac{n+1}{n+2} \times$$

$$A - \frac{m+1}{m+2} \times \text{fin. } 2A + \frac{m+1}{m+2} \times \frac{m}{m+3} \times \frac{1}{2} \text{fin. } 4A, \text{ \&c. Whence, by adding the fluents of both terms together, we have, after}$$

$$\text{proper reduction } \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n \cdot n+2}$$

$$\times \left\{ A - \frac{m-1}{m+2} \times \text{fin. } 2A + \frac{m \cdot m-7}{m+2 \cdot m+3} \times \frac{1}{2} \text{fin. } 4A - \frac{m \cdot m-1 \cdot m-17}{m+2 \cdot m+3 \cdot m+4} \right. \\ \left. \times \frac{1}{3} \text{fin. } 6A + \frac{m \cdot m-1 \cdot m-2 \cdot m-31}{m+2 \cdot m+3 \cdot m+4 \cdot m+5} \times \frac{1}{4} \text{fin. } 8A, \text{ \&c.} \right.$$

where the law of continuation is manifest, the differences (6, 10, 14, &c.) of the numbers 1, 7, 17, 31, &c. being in arithmetical progression.

COROLLARY III.

Moreover, from hence the fluent of $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} \times e + fx^2 + gx^4 + bx^6$

$$\text{\&c. may be easily deduced: for, putting } \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \dots \frac{n-1}{n}$$

$= q$, the fluent of the first term $\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} \times e$, is given, by Co-

$$\text{rol. } I. = qe \times A - \frac{m}{m+1} \times \sin. 2A + \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{1}{2} \sin. 4A \&c.$$

and that of the second term $\left(\frac{x^{\frac{n}{2}}}{\sqrt{1-xx}} \times f x^2 \right) = \frac{n+1}{n+2} \times qf \times$

$$A - \frac{m+1}{m+2} \times \sin. 2A + \frac{m+1}{m+2} \times \frac{m}{m+3} \times \frac{1}{2} \sin. 4A \&c. \text{ There-}$$

fore the fluent of the whole expression will be found $= q$

$$\times \left\{ \begin{array}{l} e \times A - \frac{m}{m+1} \times \sin. 2A + \frac{m}{m+1} \times \frac{m-1}{m+2} \times \frac{1}{2} \sin. 4A \&c. \\ \frac{n+1}{n+2} f \times A - \frac{m+1}{m+2} \times \sin. 2A + \frac{m+1}{m+2} \times \frac{m}{m+3} \times \frac{1}{2} \sin. 4A \&c. \\ \frac{n+1}{n+2} \times \frac{n+3}{n+4} g \times A - \frac{m+2}{m+3} \times \sin. 2A + \frac{m+2}{m+3} \times \frac{m+1}{m+4} \times \frac{1}{2} \sin. 4A \&c. \end{array} \right.$$

$\&c.$ $\&c.$

which, by making $r = \frac{n+1}{n+2} \times q$, $s = \frac{n+3}{n+4} \times r$, $t = \frac{n+5}{n+6} \times s$,

$\&c.$ will be reduced to

$$\left\{ \begin{array}{l} + eq + fr + gs + \&c. \times A \\ - \frac{m}{m+1} \cdot eq + \frac{m+1}{m+2} \cdot fr + \frac{m+2}{m+3} \cdot gs + \&c. \times \sin. 2A \\ + \frac{m}{m+1} \cdot \frac{m-1}{m+2} \cdot eq + \frac{m+1}{m+2} \cdot \frac{m}{m+3} \cdot fr + \frac{m+2}{m+3} \cdot \frac{m+1}{m+4} \cdot gs, \&c. \times \frac{1}{2} \sin. 4A \\ - \frac{m}{m+1} \cdot \frac{m-1}{m+2} \cdot \frac{m-2}{m+3} \cdot eq + \frac{m+1}{m+2} \cdot \frac{m}{m+3} \cdot \frac{m-1}{m+4} \cdot fr + \&c. \times \frac{1}{2} \sin. 6A \end{array} \right.$$

$\&c.$ $\&c.$

SCHOLIUM.

From the fluents determined in the preceding Proposition, those of $\frac{z^{mp+\frac{1}{2}p-1}z}{\sqrt{a-bz^p}}$, $\sqrt{a-bz^p} \times z^{mp+\frac{1}{2}p-1}z$, and $\frac{z^{mp+\frac{1}{2}p-1}z}{\sqrt{a-bz^p}}$

$\times e + fz^p + gz^{2p} + bz^{3p} \&c.$ (where m denotes any whole positive number, and p any positive number whatever, whole or broken) may be easily deduced, by means of a proper transformation:

mation : for, $\sqrt{a - bz^p}$ being $= a^{\frac{1}{2}} \sqrt{1 - \frac{bz^p}{a}}$, let there be made $\frac{bz^p}{a} = x^2$, or $z^p = \frac{a}{b} x^2$; then $z^{mp+\frac{1}{2}p} = \frac{a^{\frac{m+\frac{1}{2}}{b}}}{b^{\frac{m+\frac{1}{2}}{b}}} \times x^{2m+1}$; and consequently, by taking the fluxion on both sides, $\overline{mp + \frac{1}{2}p} \cdot z^{mp+\frac{1}{2}p-1} \dot{z} = \frac{a^{\frac{m+\frac{1}{2}}{b}}}{b^{\frac{m+\frac{1}{2}}{b}}} \times 2m+1 \cdot x^{2m} \dot{x}$, or $z^{mp+\frac{1}{2}p-1} \dot{z} = \frac{a^{\frac{m+\frac{1}{2}}{b}}}{b^{\frac{m+\frac{1}{2}}{b}}} \times \frac{2}{p} \cdot x^{2m} \dot{x}$. Therefore our first expression, $\frac{z^{mp+\frac{1}{2}p-1} \dot{z}}{\sqrt{a - bz^p}}$,

will be transformed to $\frac{2a^m}{pb^m + \frac{1}{2}} \times \frac{x^{2m} \dot{x}}{\sqrt{1-x^2}}$ (supposing $n = 2m$); whose fluent (by Problem II. Corollary I.) will be given = $\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n} \times \frac{2a^m}{pb^m + \frac{1}{2}} \times$

$A - \frac{m}{m+1} \text{fin. } 2A + \frac{m}{m+1} \cdot \frac{m-1}{m+2} \times \frac{1}{2} \text{fin. } 4A - \frac{m}{m+1} \cdot \frac{m-1}{m+2} \cdot \frac{m-2}{m+3} \times \frac{1}{3} \text{fin. } 6A + \mathcal{E}c.$ where A represents the arch whose sine is $\sqrt{\frac{bz^p}{a}}$, the radius being unity.

In the same manner the fluent of our second expression, $\sqrt{a - bz^p} \times z^{mp+\frac{1}{2}p-1} \dot{z}$, will be given (by Corol. II.) equal to $\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n+2} \times \frac{2a^{m+\frac{1}{2}}}{pb^{m+\frac{1}{2}}}$

$\times \left\{ A - \frac{m-1}{m+2} \text{fin. } 2A + \frac{m \cdot m-1}{m+2 \cdot m+3} \times \frac{1}{2} \text{fin. } 4A - \frac{m \cdot m-1 \cdot m-2}{m+2 \cdot m+3 \cdot m+4} \right.$
 $\left. \times \frac{1}{3} \text{fin. } 6A + \frac{m \cdot m-1 \cdot m-2 \cdot m-3}{m+2 \cdot m+3 \cdot m+4 \cdot m+5} \times \frac{1}{4} \text{fin. } 8A - \mathcal{E}c. \right.$

Lastly, $\frac{z^{mp+\frac{1}{2}p-1} \dot{z}}{\sqrt{a - bz^p}} \times e + f \dot{z}^p + g z^{2p} + h z^{3p} \mathcal{E}c.$ will be trans-

formed to $\frac{2a^m}{pb^m + \frac{1}{2}} \times \frac{x^{2m} \dot{x}}{\sqrt{1-x^2}} \times e + \frac{fa}{b} x^2 + \frac{ga^2}{bb} x^4 + \mathcal{E}c.$ and the fluent thereof (by Corol. III.) will therefore be given equal to

$\frac{2a^m}{pb^m + \frac{1}{2}} \times \left\{ +eq + fr \times \frac{a}{b} + gs \times \frac{a^2}{b^2} + \mathcal{E}c. \times A \right.$
 $\left. - eq \times \frac{m}{m+1} + fr \times \frac{a}{b} \times \frac{m+1}{m+2} + gs \times \frac{a^2}{b^2} \times \frac{m+2}{m+3} \mathcal{E}c. \times \text{fin. } 2A \right.$

$$\frac{2a^m}{pb^m + \frac{1}{2}} \times \frac{eq \times \frac{m}{m+1} \times \frac{m-1}{m+2} + fr \times \frac{a}{b} \times \frac{m+1}{m+2} \times \frac{m}{m+3}}{\mathcal{E}c.} \times \frac{m}{m+3} \mathcal{E}c. \times \frac{1}{2} f. 4A$$

wherein $q = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n}$, $r = q \times \frac{n+1}{n+2}$, $s = r \times \frac{n+3}{n+4}$,
 $t = s \times \frac{n+5}{n+6}$, $\mathcal{E}c.$

When $\sqrt{\frac{bz^n}{a}}$ becomes equal to the radius, or unity, A will be an arch of 90 degrees; and therefore, the sines of all the arches 2A, 4A, 6A, &c. being then equal to nothing, the fluent of $\frac{z^{mp + \frac{1}{2}p - 1} z}{\sqrt{a - bz^n}}$ will, in that circumstance, be barely = $\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n-2} \times \frac{2a^m}{pb^m + \frac{1}{2}} \times A$. Moreover, the fluent of $\sqrt{a - bz^n} \times z^{mp + \frac{1}{2}p - 1} z$ will then become $\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots n-1}{2 \cdot 4 \cdot 6 \cdot 8 \dots n+2} \times \frac{2a^m + 1}{pb^m + \frac{1}{2}} \times A$; and that of $\frac{z^{mp + \frac{1}{2}p - 1} z}{\sqrt{a - bz^n}} \times e + fz^p + gz^{2p} + bz^{3p}$ $\mathcal{E}c. = e + \frac{n+1}{n+2} \cdot \frac{fa}{b} + \frac{n+1}{n+2} \cdot \frac{n+3}{n+4} \cdot \frac{ga^2}{b^2} + \frac{n+1}{n+2} \cdot \frac{n+3}{n+4} \cdot \frac{n+5}{n+6} \cdot \frac{ba^3}{b^3} \mathcal{E}c.$ $\times \frac{2qa^m}{pb^m + \frac{1}{2}} \times A$; where $q = \frac{1 \cdot 3 \cdot 5 \dots n-1}{2 \cdot 4 \cdot 6 \dots n}$, and where, if $m = 0$, q must be taken = 1.

PROBLEM III.

To determine the fluent of $z \times \text{cof. } mz \times \text{cof. } nz \times \text{cof. } pz$ &c. in which mz , nz , pz , &c. are any given multiples of the arch z ; the radius of the circle being unity.

Make $A = mz$, $B = nz$, $C = pz$, $\mathcal{E}c.$ and find (by the method on p. 81) a series of co-sines of the multiples of z , to express the continual product ($\text{cof. } A \times \text{cof. } B \times \text{cof. } C$, $\mathcal{E}c.$) of the co-sines propounded; which series let be denoted by $a \times \text{cof. } az + \text{cof. } \beta z + \text{cof. } \gamma z + \text{cof. } \delta z$ $\mathcal{E}c.$ (a , β , $\mathcal{E}c.$ being constant quantities): then will our given expression become az

$a\dot{z} \times \text{cof. } az + \text{cof. } \beta z + \text{cof. } \gamma z + \text{cof. } \delta z \mathcal{E}c.$ and its fluent will therefore (by proceeding as in Prob. II.) appear to be $= a$ into $\frac{\text{fin. } az}{a} + \frac{\text{fin. } \beta z}{\beta} + \frac{\text{fin. } \gamma z}{\gamma} + \frac{\text{fin. } \delta z}{\delta} \mathcal{E}c.$

Thus, for example, let the fluent of $\dot{z} \times \text{cof. } mz \times \text{cof. } nz$ be required; then will $\text{cof. } A \times \text{cof. } B (= \text{cof. } mz \times \text{cof. } nz) = \frac{1}{2} \times \text{cof. } A + B + \text{cof. } A - B = \frac{1}{2} \times \text{cof. } \overline{m+n} . z + \overline{m-n} . z$: therefore, in this case, $a = \frac{1}{2}$, $\alpha = m + n$, and $\beta = m - n$; and consequently the fluent sought $= \frac{1}{2}$ into $\frac{\text{fin. } \overline{m+n} . z}{m+n} + \frac{\text{fin. } \overline{m-n} . z}{m-n}$.

In like manner, if the fluent of $\dot{z} \times \text{cof. } mz \times \text{cof. } nz \times \text{cof. } pz$ were to be required; then would $\text{cof. } mz \times \text{cof. } nz \times \text{cof. } pz (= \text{cof. } A \times \text{cof. } B \times \text{cof. } C) = \frac{1}{4}$ into $\text{cof. } \overline{m+n+p} . z + \text{cof. } \overline{m+n-p} . z + \text{cof. } \overline{m-n+p} . z + \text{cof. } \overline{-m+n+p} . z$; and therefore the fluent sought $= \frac{1}{4}$ into $\frac{\text{fin. } \overline{m+n+p} . z}{m+n+p} + \frac{\text{fin. } \overline{m+n-p} . z}{m+n-p} + \frac{\text{fin. } \overline{m-n+p} . z}{m-n+p} + \frac{\text{fin. } \overline{-n+m+p} . z}{-n+m+p}$.

By the very same method the fluent may be determined when some, or all of the factors are fines (instead of co-fines). Thus, if there be given $\dot{z} \times \text{cof. } mz \times \text{fin. } nz$, it may be wrote $\dot{z} \times \text{cof. } mz \times \text{cof. } 90^\circ - nz$; which is $= \frac{\dot{z}}{2}$ into $\text{cof. } 90^\circ + \overline{m-n} . z + \text{cof. } \overline{m+n} . z - 90^\circ = \frac{\dot{z}}{2}$ into $\text{fin. } \overline{n-m} . z + \text{fin. } \overline{n+m} . z$; and so the fluent (by proceeding as in Problem I.) will come out $= \frac{1}{2}$ into $\frac{\text{versed-fin. } \overline{n-m} . z}{n-m} + \frac{\text{versed-fin. } \overline{n+m} . z}{n+m}$.

PROBLEM IV.

From the equation $ay + \frac{by}{z} + \frac{cy}{z^2} + \frac{dy}{z^3} + \frac{ey}{z^4} + \&c. = 0$
 (wherein $a, b, c, d, \&c.$ denote constant quantities); it is proposed to find the value of y in terms of z .

Assume $y = \alpha M^{mz} + \beta M^{nz} + \gamma M^{pz} + \delta M^{qz} \&c.$ in which M denotes the number whose hyperbolic logarithm is unity: then will $\dot{y} = m\dot{z}\alpha M^{mz} + n\dot{z}\beta M^{nz} + p\dot{z}\gamma M^{pz} \&c.$
 $\ddot{y} = m^2\dot{z}^2\alpha M^{mz} + n^2\dot{z}^2\beta M^{nz} + p^2\dot{z}^2\gamma M^{pz} \&c.$
 $\ddot{\dot{y}} = m^3\dot{z}^3\alpha M^{mz} + n^3\dot{z}^3\beta M^{nz} + p^3\dot{z}^3\gamma M^{pz} \&c.$
 $\&c. \qquad \&c.$

Which values being substituted in the given equation, we have

$$\left. \begin{array}{l} a\alpha M^{mz} + a\beta M^{nz} + a\gamma M^{pz} \&c. \\ b m \alpha M^{mz} + b n \beta M^{nz} + b p \gamma M^{pz} \&c. \\ c m^2 \alpha M^{mz} + c n^2 \beta M^{nz} + c p^2 \gamma M^{pz} \&c. \\ d m^3 \alpha M^{mz} + d n^3 \beta M^{nz} + d p^3 \gamma M^{pz} \&c. \end{array} \right\} = 0$$

$\&c. \qquad \&c.$

From whence, by equating the homologous terms, we have $a + bm + cm^2 + dm^3 \&c. = 0$, $a + bn + cn^2 + dn^3 \&c. = 0$, $a + bp + cp^2 + dp^3 \&c. = 0$, $\&c.$ that is, the required values of $m, n, p, \&c.$ will always be the roots of an equation, $a + bx + cx^2 + dx^3 \&c. = 0$, wherein the given quantities are the same, in every term, with those in the fluxional equation propounded. Therefore, when these roots are known, the value of y will also be known: in which the coefficients $\alpha, \beta, \gamma, \delta, \&c.$ may denote any constant quantities at pleasure; as is evident from the process.

When some of the roots of the equation $a + bx + cx^2 + dx^3 + ex^4 \&c. = 0$, happen to be impossible, the values of the corresponding terms of the series $\alpha M^{mz} + \beta M^{nz} + \gamma M^{pz} + \delta M^{qz} \&c.$ will then be expressed by means of the sines and co-sines of circular arcs. Thus, for example, let the fluxionary equation propounded be $y - \frac{\dot{y}}{z^4} = 0$; then we shall have

$1 - x^4 = 0$; whereof the four roots are $1, -1, +\sqrt{-1}$, and $-\sqrt{-1}$; and, these being substituted for m, n, p , and

q , respectively, y will here become $= \alpha M^z + \beta M^{-z} + \gamma M^{+z\sqrt{-1}} + \delta M^{-z\sqrt{-1}}$. Now, to take away the imaginary terms $\gamma M^{+z\sqrt{-1}} + \delta M^{-z\sqrt{-1}}$, we may write $k + l = \gamma$, and $k - l = \delta$; whereby the sum of the said terms will be $= k \times M^{z\sqrt{-1}} + M^{-z\sqrt{-1}} + l \times M^{z\sqrt{-1}} - M^{-z\sqrt{-1}} = 2k \times \text{cof. } z + \frac{2l}{\sqrt{-1}} \times \text{fin. } z$ (*vid. p. 78*): whence (putting $b = \frac{l}{\sqrt{-1}}$) we have $y = \alpha M^z + \beta M^{-z} + 2k \times \text{cof. } z + 2b \times \text{fin. } z$; where α , β , b , and k denote any constant quantities, at pleasure.

In like manner, supposing the equation given to be $y + \frac{dy}{z^2} = 0$, we shall have $1 + dx^3 = 0$; whereof the three roots are $-d^{-\frac{1}{3}}$, $d^{-\frac{1}{3}} \times \frac{1}{2} + \sqrt{-\frac{3}{4}}$, and $d^{-\frac{1}{3}} \times \frac{1}{2} - \sqrt{-\frac{3}{4}}$; which, if s be put $= d^{-\frac{1}{3}}$, and $t = d^{-\frac{1}{3}} \times \sqrt{\frac{3}{4}}$, will be more commodiously expressed by $-s$, $\frac{1}{2}s + t\sqrt{-1}$, and $\frac{1}{2}s - t\sqrt{-1}$: And these values being substituted in the room of m , n , and p , we shall have $y = \alpha M^{-sz} + \beta M^{\frac{1}{2}sz + tz\sqrt{-1}} + \gamma M^{\frac{1}{2}sz - tz\sqrt{-1}} = \alpha M^{-sz} + M^{\frac{1}{2}sz} \times \beta M^{tz\sqrt{-1}} + \gamma M^{-tz\sqrt{-1}}$; which, by reasoning as in the preceding case, is reduced to $y = \alpha M^{-sz} + M^{\frac{1}{2}sz} \times 2b \times \text{fin. } tz + 2k \times \text{cof. } tz$.

PROBLEM V.

From the equation $ay + \frac{by}{z} + \frac{cy}{z^2} + \frac{dy}{z^3} \mathcal{E}c. = AM^{pz} + BM^{qz} + CM^{rz} \mathcal{E}c.$ to determine the value of y ; supposing M to denote the number whose hyperbolical logarithm is unity, and a , b , c , $\mathcal{E}c.$ A , B , C , $\mathcal{E}c.$ any constant quantities.

Assuming $y = PM^{pz} + QM^{qz} + RM^{rz} \mathcal{E}c.$ we have

$$\dot{y} = p\dot{z}PM^{pz} + q\dot{z}QM^{qz} + r\dot{z}RM^{rz} \mathcal{E}c.$$

$$\ddot{y} = p^2\dot{z}^2BM^{pz} + q^2\dot{z}^2QM^{qz} + r^2\dot{z}^2RM^{rz} \mathcal{E}c.$$

$\mathcal{E}c.$

$\mathcal{E}c.$

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which

which values being substituted in the given equation, it becomes

$$\left. \begin{array}{l} aPM^{pz} + aQM^{qz} + aRM^{rz} \mathcal{E}c. \\ bpPM^{pz} + bqQM^{qz} + brRM^{rz} \mathcal{E}c. \\ cpPM^{pz} + cqQM^{qz} + crRM^{rz} \mathcal{E}c. \end{array} \right\} = AM^{pz} + BM^{qz} + CM^{rz} \mathcal{E}c.$$

From whence, by comparing the homologous terms, we have

$$P = \frac{A}{a + bp + cp^2 + dp^3 \mathcal{E}c.}, \quad Q = \frac{B}{a + bq + cq^2 + dq^3 \mathcal{E}c.}, \quad R = \frac{C}{a + br + cr^2 + dr^3 \mathcal{E}c.}, \quad \mathcal{E}c. \text{ whereby one value of } y \text{ is known.}$$

But the value or fluent thus found, in order to render it general, must be corrected by the value of y found in the preceding Problem, that is, by the quantity $\alpha M^{m'z} + \beta M^{n'z} + \gamma M^{p'z} \mathcal{E}c.$ wherein $m', n', p', \mathcal{E}c.$ denote the roots of the equation $a + bx + cx^2 + dx^3 \mathcal{E}c. = 0$, and $\alpha, \beta, \gamma, \mathcal{E}c.$ any constant quantities. For, since all the terms arising from this last part of the value of y , by substituting in the given equation, do mutually destroy one another, the other terms affected with $P, Q, R, \mathcal{E}c.$ will be no-ways influenced thereby, but remain exactly the same as above determined.

COROLLARY.

If the equation given be $m^2y + \frac{y}{zz} = AM^{pz} + BM^{qz} + CM^{rz} + DM^{sz} \mathcal{E}c.$ then (a being $= m^2$, $c = 1$, and $b, d, e, \mathcal{E}c.$ each $= 0$) we have $P = \frac{A}{mm + pp}$, $Q = \frac{B}{mm + qq}$, $\mathcal{E}c.$ and $x = \pm m\sqrt{-1}$; and consequently $y = \alpha M^{mz\sqrt{-1}} + \beta M^{-mz\sqrt{-1}} + \frac{AM^{pz}}{mm + pp} + \frac{BM^{qz}}{mm + qq} \mathcal{E}c. = 2b \times \sin. mz + 2k \times \cos. mz + \frac{AM^{pz}}{mm + pp} + \frac{BM^{qz}}{mm + qq} \mathcal{E}c. \text{ (see Ex. I. to Prob. V.)}$

Hence it follows, that, if the equation given be $m^2y + \frac{y}{zz} (= AM^{pz\sqrt{-1}} + AM^{-pz\sqrt{-1}} + A'M^{pz\sqrt{-1}} + A'M^{-pz\sqrt{-1}} \mathcal{E}c.) = 2A \times \cos. \pi z + 2A' \times \cos. \varphi z \mathcal{E}c.$ the value of y (by substituting $-\pi^2 = p^2$, $-\pi^2 = q^2$, $-\varphi^2 = r^2$, $-\varphi^2 = s^2$, $A = B$, $A' = C$, $A' = D$, $\mathcal{E}c.$) will come out $= 2b \times \sin. mz$

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$$+ 2k \times \text{cof. } mz + \frac{AM^{\pi z \sqrt{-1}}}{mm - \pi\pi} + \frac{AM^{-\pi z \sqrt{-1}}}{mm - \pi\pi} \mathcal{E}c. = 2b \times \text{fin. } mz$$

$$+ 2k \times \text{cof. } mz + \frac{2A}{mm - \pi\pi} \times \text{cof. } \pi z + \frac{2A'}{mm - \pi\pi} \times \text{cof. } \rho z \mathcal{E}c.$$

Which equation (wherein b and k may denote any constant quantities) is of singular use in determining the figure of the lunar orbit.

In like manner, when the general equation propounded is of this form, $ay + \frac{by}{z} + \frac{cy}{z^2} + \frac{dy}{z^3} \mathcal{E}c. = Az^v + Bz^{v-1} + Cz^{v-2} + Dz^{v-3} \mathcal{E}c.$ the value of y may be determined, by assuming $Pz^v + Qz^{v-1} + Rz^{v-2} \mathcal{E}c. = y$; from whence, by substituting in the given equation, and comparing the homologous terms, there will be had $P = \frac{A}{a}$, $Q = \frac{B - v b P}{a}$, $R = \frac{C - v-1 \cdot b Q - v \cdot v-1 \cdot c P}{a}$, $S = \frac{D - v-2 \cdot b R - v-1 \cdot v-2 \cdot c Q - v \cdot v-1 \cdot v-2 \cdot d P}{a}$, $\mathcal{E}c.$ where the series will always terminate, provided v is any positive integer; and where, if to the value of y thus determined, the correction or series ($\alpha M^{mz} + \beta M^{\pi z} + \gamma M^{\rho z} \mathcal{E}c.$) found by Prob. IV. be added, the general value of y will be obtained.

PROBLEM VI.

To determine the value of y in any fluxionary equation of this form, $\frac{\ddot{y}}{z^4} + \frac{\dot{a}y}{z^3} + \frac{by}{z^2} + \frac{cy}{z} + dy = \Delta$; supposing Δ to represent any quantity expressed in terms of z and known coefficients.

1°. Make $y = M^{pz} \times \text{flu. } \dot{z} P M^{-pz}$ (wherein P denotes a variable quantity, and p a constant one, to be determined): so shall $y \times M^{-pz} = \text{flu. } \dot{z} P M^{-pz}$, or (by taking the fluxions) $\dot{y} \times M^{-pz} = y \times p \dot{z} M^{-pz} = \dot{z} P M^{-pz}$; whence, dividing the whole by $\dot{z} M^{-pz}$, we have $\frac{\dot{y}}{z} - p y = P$.

2°. Make $P = M^{qz} \times \text{flu. } \dot{z} Q M^{-qz}$; then our last equation will be transformed to $\frac{\dot{y}}{z} - p y \times M^{-qz} = \text{flu. } \dot{z} Q M^{-qz}$;
whence,

whence, by taking the fluxions, $\frac{\ddot{y}}{z} - p\dot{y} \times M^{-rz} + \frac{\dot{y}}{z} - py \times -q\dot{z}M^{-rz} = \dot{z}QM^{-rz}$, or $\frac{\ddot{y}}{z} - \overline{p+q} \cdot \frac{\dot{y}}{z} + pqy = \mathcal{Q}$, by dividing the whole by $\dot{z}M^{-rz}$.

3°. Make, now, $Q = M^{-rz} \times \text{flu. } \dot{z}RM^{-rz}$; then will $\frac{\ddot{y}}{z^2} - \overline{p+q} \cdot \frac{\dot{y}}{z} + pqy \times M^{-rz} = \text{fluent of } \dot{z}RM^{-rz}$, or $\frac{\dot{y}}{z^2} - \overline{p+q} \cdot \frac{\dot{y}}{z} + pq\dot{y} \times M^{-rz} + \frac{\ddot{y}}{z^2} - \overline{p+q} \cdot \frac{\dot{y}}{z} + pqy \times -r\dot{z}M^{-rz} = \dot{z}RM^{-rz}$, or, lastly, $\frac{\ddot{y}}{z^2} - \overline{p+q+r} \cdot \frac{\dot{y}}{z} + \overline{pq+pr+qr} \cdot \frac{\dot{y}}{z} - pqry = R$.

4°. Make, again, $R = M^{-rz} \times \text{flu. } \dot{z}SM^{-rz}$, and proceed in the same manner; so shall $\frac{\ddot{y}}{z^3} - \overline{p+q+r+s} \cdot \frac{\dot{y}}{z^2} - \overline{pq+pr+ps+qr+qs+rs} \cdot \frac{\dot{y}}{z} - \overline{pqr+pqrs+prqs+qrs} \cdot \frac{\dot{y}}{z} + pqr sy = S$: from whence the law of continuation is manifest.

Let, now, the several terms of the equation $\frac{\ddot{y}}{z^4} - \overline{p+q+r+s} \cdot \frac{\dot{y}}{z^3} \&c. = S$ be compared with the corresponding terms of the given one, $\frac{\ddot{y}}{z^4} + \frac{a\dot{y}}{z^3} \&c. = \Delta$: so shall $p+q+r \&c. = -a$, $pq+pr+ps+qr \&c. = -b$, $pqr+pqrs \&c. = -c$, $\&c. \&c.$ Whence, from the *genesis* of equations, it is evident, that $p, q, r, \&c.$ are the roots of an equation $x^4 + ax^3 + bx^2 + cx + d = 0$ (or, $x^4 + ax^{n-1} + bx^{n-2} \&c. = 0$) wherein the given quantities are the very same with those in the equation propounded. Therefore, when the values of these roots are found (by any of the known methods) the values of R, Q, P , and y may also be found, one from another, successively. $\mathcal{Q}, E. I.$

The same otherwise.

Let (if possible) $y = AM^{p^z} \times \text{flu. } \dot{z}\Delta M^{-p^z} + BM^{q^z} \times \text{flu. } \dot{z}\Delta M^{-q^z} + CM^{r^z} \times \text{flu. } \dot{z}\Delta M^{-r^z} \&c.$ (A, B, C, $\&c.$ $p, q, r, \&c.$ being constant quantities to be determined): then, by taking the fluxions, we shall have

$$\frac{\dot{y}}{z} = pAM^{p^z} \times \text{flu. } \dot{z}\Delta M^{-p^z} + A\Delta + qBM^{q^z} \times \text{flu. } \dot{z}\Delta M^{-q^z} + \&c.$$

$$\frac{\ddot{y}}{z^2} = p^2 AM^{p^z} \times \text{flu. } \dot{z}\Delta M^{-p^z} + pA\Delta + \frac{A\dot{\Delta}}{z} + q^2 BM^{q^z} \times \text{flu. } \dot{z}\Delta M^{-q^z} \&c.$$

$$\frac{\ddot{\dot{y}}}{z^3} = p^3 AM^{p^z} \times \text{flu. } \dot{z}\Delta M^{-p^z} + p^2 A\Delta + \frac{pA\dot{\Delta}}{z} + \frac{A\ddot{\Delta}}{z^2} + \&c.$$

$$\frac{\ddot{\dot{\dot{y}}}}{z^4} = p^4 AM^{p^z} \times \text{flu. } \dot{z}\Delta M^{-p^z} + p^3 A\Delta + \frac{p^2 A\dot{\Delta}}{z} + \frac{pA\ddot{\Delta}}{z^2} + \frac{A\ddot{\dot{\Delta}}}{z^3} \&c.$$

Which values being substituted in the given equation, and the homologous terms being compared, we shall thereby get $p^4 + ap^3 + bp^2 + cp + d = 0$, $q^4 + aq^3 + bq^2 + cq + d = 0$, $\&c.$

$$\text{also } \left\{ \frac{p^3 + ap^2 + bp + c \times A + q^3 + aq^2 + bq + c \times B}{r^3 + ar^2 + br + c \times C + s^3 + as^2 + bs + c \times D} \right\} = 1,$$

$$\frac{p^3 + ap^2 + bp + c \times A + q^3 + aq^2 + bq + c \times B}{p + a \times A + q + a \times B + r + a \times C + s + a \times D} = 0$$

$$A + B + C + D = 0.$$

Now, from the former of these equations, $p^4 + ap^3 + bp^2 + cp + d = 0$, $q^4 + aq^3 + bq^2 + cq + d = 0$, $\&c.$ it appears evident, that $p, q, r, \&c.$ are the roots of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ (or, more generally, of $x^n + ax^{n-1} + bx^{n-2} \&c. = 0$, n denoting the order to which the fluxions ascend in the given equation); which roots being therefore found (by any of the known methods) the values of $p, q, r, \&c.$ will be obtained. But to find from thence and the remaining equations, the values of A, B, C, $\&c.$ let the last of these equations multiplied by a , be subtracted from the preceding one, so shall $pa + qB + rC + sD = 0$: moreover, let this new equation multiplied by a , be subtracted from the last but two, and from the remainder let $bA + bB + bC + bD = 0$ be again subtracted, whence will be had $p^2A + q^2B$
+

$+r^2C + s^2D = 0$: and, in the same manner, from the first equation, will be had $p^3A + q^3B + r^3C + s^3D = 1$ (because 1, and not 0, forms the latter part of that equation).

Now, from each of the equations ($A + B + C + D = 0$, $pA + qB + rC + sD = 0$, $p^2A + q^2B + r^2C + s^2D = 0$, $p^3A + q^3B + r^3C + s^3D = 1$) thus derived, let the preceding one multiplied by p , be subtracted: so shall

$$\overline{q-p} \cdot B + \overline{r-p} \cdot C + \overline{s-p} \cdot D = 0,$$

$$\overline{q-p} \cdot qB + \overline{r-p} \cdot rC + \overline{s-p} \cdot sD = 0,$$

$$\overline{q-p} \cdot q^2B + \overline{r-p} \cdot r^2C + \overline{s-p} \cdot s^2D = 1.$$

Moreover, from each of these last equations, let the preceding one multiplied by q , be in like manner subtracted; whence will be had

$$\overline{r-p} \cdot \overline{r-q} \cdot C + \overline{s-p} \cdot \overline{s-q} \cdot D = 0,$$

$$\overline{r-p} \cdot \overline{r-q} \cdot rC + \overline{s-p} \cdot \overline{s-q} \cdot sD = 1.$$

Again, from the last of these, let the preceding one multiplied by r , be subtracted; then will $\overline{s-p} \cdot \overline{s-q} \cdot \overline{s-r} \cdot D = 1$, and consequently $D = \frac{1}{\overline{s-p} \cdot \overline{s-q} \cdot \overline{s-r}}$: whence it is manifest by inspection (because p is the same with respect to A , as s is to D , &c.) that $A = \frac{1}{\overline{p-q} \cdot \overline{p-r} \cdot \&c.}$, $B = \frac{1}{\overline{q-p} \cdot \overline{q-r} \cdot \&c.}$,

$C = \frac{1}{\overline{r-p} \cdot \overline{r-q} \cdot \&c.}$, &c. From whence the value of $y (=$

$AM^{px} \times \text{flu.} \dot{z} \Delta M^{-px} + BM^{qx} \times \text{flu.} \dot{z} \Delta M^{-qx} \&c.)$ will be known, let the orders of fluxions in the equation ascend to what height they will. — Thus, for example, let the equation propounded be $\frac{y}{z^2} + m^2y = M^{px}$: in which case, a being $= 0$, $b = m^2$,

$c = 0$, &c. our general equation, $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3}$ &c. $= 0$, will therefore become $x^2 + m^2 = 0$; whereof the two roots (p and q) are $m\sqrt{-1}$, and $-m\sqrt{-1}$; from whence $A (= \frac{1}{p-q}) = \frac{1}{2m\sqrt{-1}}$, and $B (= \frac{1}{q-p}) = -$

$\frac{1}{2m\sqrt{-1}}$: also, because Δ is here $= M^{px}$, we have $y = AM^{px}$

$$\times \text{flu. } \dot{z}M^{\pi z - pz} + BM^{qz} \times \text{flu. } \dot{z}M^{\pi z - qz} = \frac{AM^{\pi z}}{\pi - p} + \frac{BM^{\pi z}}{\pi - q} =$$

$\frac{M^{\pi z}}{\pi^2 + p^2}$ (by substituting the values of p , q , A , and B). But

in order to render the solution general, the value of y thus found must, always, be corrected, or augmented by the quantity $\alpha M^{pz} + \beta M^{qz} + \gamma M^{rz}$ &c. (given by Prob. IV.) where α , β , γ , δ , &c. may denote any constant quantities whatever, positive, or negative.—Other instances of the use of exponential quantities, and of the measures of angles and ratios, in the resolution of fluxionary equations of different kinds, might be given; but I shall conclude here, with observing, that Δ , in this last solution, may denote any quantity wherein both y and z enter, as well as one in which z is alone concerned independent of y .



An INVESTIGATION of a GENERAL RULE for
the Resolution of Isoperimetrical Problems of all
Orders.

L E M M A.

UPPOSING $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ to be a series of in-
determinate quantities,

and that $\left\{ \begin{array}{l} \mathcal{Q}, R, S, T \\ \mathcal{Q}', R', S', T' \\ \mathcal{Q}'', R'', S'', T'' \\ \mathcal{Q}''', R''', S''', T''' \\ \mathcal{Q}'''' , R'''' , S'''' , T'''' \\ \&c. \end{array} \right\}$ are any quantities composed of $\left\{ \begin{array}{l} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \&c. \end{array} \right\}$ and given quantities;

It is proposed to find an equation for the relation of $\alpha, \beta, \gamma, \delta, \epsilon, \&c.$ so that the quantity $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}''' + \mathcal{Q}'''' \&c.$ shall be a maximum or minimum, at the same time that the other quantities $R + R' + R'' + R''' + R'''' \&c.$ $S + S' + S'' + S''' + S'''' \&c.$ and $T + T' + T'' + T''' + T'''' \&c.$ are all of them given, or supposed to remain invariable.

Let $\bar{\mathcal{Q}}, \bar{R}, \bar{S}, \bar{T}$ denote any corresponding terms of the series's $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}''' + \mathcal{Q}'''' \&c.$ $R + R' + R'' + R''' + R'''' \&c.$ $S + S' + S'' + S''' + S'''' \&c.$ $T + T' + T'' + T''' + T'''' \&c.$ respectively, expressed in terms of u , any one of the proposed quantities $\alpha, \beta, \gamma, \delta, \epsilon \&c.$ moreover let the fluxion of $\mathcal{Q}$ (α being variable) be denoted by $q\dot{\alpha}$; that of R by $r\dot{\alpha}$; that of $\mathcal{Q}'$ by $q'\dot{\beta}$; that R' by $r'\dot{\beta}$, $\&c. \&c.$

It is evident that the quantity $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}''' + \mathcal{Q}'''' \&c.$ cannot be a maximum or minimum, when $R + R' + R'' + R''' + R'''' \&c.$ $S + S' + S'' + S''' + S'''' \&c.$ and $T + T' + T'' + T''' + T'''' \&c.$ are given quantities, unless the part $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}'''$ is a maximum or minimum, when the parts

parts $\bar{R} + R + R' + R''$, $\bar{S} + S + S' + S''$, and $\bar{T} + T + T' + T''$ are given quantities; because the terms in these parts may be alone made variable, while the other terms are supposed to remain the same, whereby the whole sums, $R + R' + R'' + R''' + R''''$ &c. $S + S' + S'' + S''' + S''''$ &c. will remain the same, and the quantity $\bar{Q} + \bar{Q}' + \bar{Q}'' + \bar{Q}''' + \bar{Q}''''$ &c. will be a *maximum* or *minimum*, when the part $\bar{Q} + \bar{Q}' + \bar{Q}'' + \bar{Q}'''$ is so. But when $\bar{Q} + \bar{Q}' + \bar{Q}'' + \bar{Q}'''$ is a *maximum* or *minimum*, and $\bar{R} + R + R' + R''$, $\bar{S} + S + S' + S''$, and $\bar{T} + T + T' + T''$ are given (or constant) quantities, their fluxions will be, all of them, equal to nothing; whence we have these equations,

$$\bar{q}\dot{u} + q\dot{\alpha} + q'\dot{\beta} + q''\dot{\gamma} = 0$$

$$\bar{r}\dot{u} + r\dot{\alpha} + r'\dot{\beta} + r''\dot{\gamma} = 0$$

$$\bar{s}\dot{u} + s\dot{\alpha} + s'\dot{\beta} + s''\dot{\gamma} = 0$$

$$\bar{t}\dot{u} + t\dot{\alpha} + t'\dot{\beta} + t''\dot{\gamma} = 0$$

In order now to exterminate the fluxions $\dot{u}$, $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$, let these equations be respectively multiplied by 1, e , f , g , (yet unknown) and let all the products thence arising be added together; whence will be had $\bar{q} + er + fs + gt \times \dot{u} + q + er + fs + gt \times \dot{\alpha} + q' + er' + fs' + gt' \times \dot{\beta} + q'' + er'' + fs'' + gt'' \times \dot{\gamma} = 0$.

$$\text{Make, now, } \bar{q} + er + fs + gt = 0,$$

$$q' + er' + fs' + gt' = 0,$$

$$q'' + er'' + fs'' + gt'' = 0,$$

From whence, there being as many equations as quantities, e , f , g , to be determined, the values of those quantities will always be given from thence, in terms of the quantities $q, r, s, t, q', r', s', t'; q'', r'', s'', t''$ (exclusive of $\bar{q}, \bar{r}, \bar{s}, \bar{t}$). Now, seeing all the terms of the equation $\bar{q} + er + fs + gt \times \dot{u} + q + er + fs + gt \times \dot{\alpha} + q' + er' + fs' + gt' \times \dot{\beta} + q'' + er'' + fs'' + gt'' \times \dot{\gamma} = 0$, after the first $(\bar{q} + er + fs + gt \times \dot{u})$ do thus actually vanish (by their coefficients being taken equal to nothing), it is evident, therefore, that $\bar{q} + er + fs + gt$ must also be $= 0$ (or *flux.* $\bar{Q} + e \text{ flux. } \bar{R} + f \times \text{flux. } \bar{S} + g \times \text{flux. } \bar{T} = 0$); where

An Investigation of a general Rule

e, f, g being quantities depending intirely upon q, r, s, t , &c. (exclusive of $\bar{q}, \bar{r}, \bar{s}, \bar{t}$), they must necessarily be invariable, or continue of the same value, let $\bar{q}, \bar{r}, \bar{s}, \bar{t}$, stand for which terms you will of the corresponding series's, $q''' + q''''$ &c. $r''' + r''''$ &c. because the quantities $q, r, s, t; \bar{q}, \bar{r}, \bar{s}, \bar{t}, q'', r'', s'', t''$, (on which e, f, g , depend) have themselves a determinate value each, in the required circumstance, when $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}''$ &c. is a maximum, or minimum.

PROPOSITION.

Supposing y and u to be two flowing quantities, and that $\bar{\mathcal{Q}}, \bar{R}, \bar{S}, \bar{T}$ &c. are quantities expressed in terms of y, u , and given coefficients; 'tis proposed to find an equation, expressing the relation of y and u (or of $\bar{\mathcal{Q}}, \bar{R}, \bar{S}, \bar{T}$, &c.) so that the fluent of $\mathcal{Q}y$, corresponding to a given value of y , shall be a maximum or minimum, and the fluents of $\bar{R}y, \bar{S}y, \bar{T}y$, &c. all of them, at the same time, equal to given quantities.

Let $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'', \mathcal{Q}'''$, &c. be the different values of $\mathcal{Q}$, that will arise, when y is, successively, expounded by the terms of a given arithmetical progression whose common difference is the indefinitely small quantity y' ($\alpha, \beta, \gamma, \delta$, &c. denoting the respective values of u), and let R, R', R'', R''' , &c. be the corresponding values of $\bar{R}$, &c. &c. Then it is well known that the sum of all the quantities $\mathcal{Q}y' + \mathcal{Q}'y' + \mathcal{Q}''y' + \mathcal{Q}'''y' + \mathcal{Q}''''y'$ &c. will be = fluent of $\bar{\mathcal{Q}}y$; and the sum of all the quantities $Ry' + R'y' + R''y' + R'''y' + R''''y'$ &c. = fluent of $\bar{R}y$, &c. But, by the Lemma, it appears, that $\mathcal{Q} + \mathcal{Q}' + \mathcal{Q}'' + \mathcal{Q}''' + \mathcal{Q}''''$ &c. or $\mathcal{Q}y' + \mathcal{Q}'y' + \mathcal{Q}''y' + \mathcal{Q}'''y' + \mathcal{Q}''''y'$ &c. (because y' is constant) will be a maximum or minimum, and the quantities $Ry' + R'y' + R''y' + R'''y'$, &c. $Sy' + S'y' + S''y' + S'''y'$, &c. at the same time equal to given ones, when the relation of y and u (or of $\bar{\mathcal{Q}}, \bar{R}, \bar{S}, \bar{T}$, &c.) is expressed by the equation, flux. $\mathcal{Q} + e \times \text{flux. } \bar{R} + f \times \text{flux. } \bar{S} + g \times \text{flux. } \bar{T} = 0$: where e, f, g , &c. denote (unknown) constant quantities; and where, in taking the fluxions of $\bar{\mathcal{Q}}, \bar{R}, \bar{S}, \bar{T}$, &c. the quantity u is, alone,

alone, to be considered as variable; because the successive values of y , entering respectively into $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{Q}'''$, &c. are constant quantities, being (by hypothesis) such as successively arise from the terms of a given arithmetical progression. Hence we have the following

GENERAL RULE

for the Resolution of Isoperimetrical Problems of all orders.

Take the fluxions of all the proposed expressions (as well that respecting the maximum or minimum, as of the others whose fluents are to be given quantities, making that quantity, and likewise its fluxion, invariable, whereof the fluxion (as well as the quantity itself) enters into the said expressions; and, having divided everywhere by the fluxion of the other quantity made variable, let the quantities hence arising, joined to general coefficients 1, e , f , g , &c. be united into one sum, and the whole be made equal to nothing: from which equation (wherein the values of e , f , g , &c. may be either positive, or negative, or nothing, as the case requires), the required relation of the two variable quantities will be truly exhibited.

To illustrate the use of the rule here laid down by an example, Fig. 22. let x and y be supposed to represent the ordinate (PQ) and abscissa (AP) of a curve $ADQE$; and suppose $AFRG$ to be another curve, having the same abscissa, whose ordinate PR is, every-where, $= ax^m y^n$; 'tis required to find the relation of x and y , so that the area $BFGC$, answering to a given value of BC , shall be a maximum or minimum, at the same time that the corresponding area $BDEC$ is equal to a given quantity. Here, by hypothesis, the fluent of $ax^m y^n$ is to be a maximum or minimum, and that of xy equal to a given quantity: taking, therefore, the fluxions of both expressions, &c. (making x alone variable, according to the rule), we thence get $max^{m-1} y^n - ey = 0$: whence $x^{m-1} y^n = \frac{e}{ma}$; and consequently $ax^m y^n (= PR) =$

$\frac{ex}{m}$. Therefore, seeing PR is in a constant ratio to PQ , it is evident, that both the curves will be of the same kind; and that they will be both hyperbolas, or both parabolas, according as the values of the exponents $m-1$, and n (in the general equation

An Investigation of a general Rule

tion $x^{m-1}y^n = \frac{e}{ma}$ are like, or unlike, with regard to *positive* and *negative*. If $m-1$ be *positive*, the equation gives a *minimum*; if *negative*, a *maximum*; but when $m-1=0$, or when $m=0$, the equation fails; in which cases there will be neither a *maximum*, nor a *minimum*.

For another example, let the fluxions given be $\frac{y\dot{x}^2}{\dot{y}}$ and $\dot{x}$; the fluent of the former (answering to a given value of y) being to be a *minimum*, and that of the latter, at the same time, equal to a given quantity. Here ($\dot{x}$ being concerned independently, either, of its fluent x or fluxion $\ddot{x}$) let the fluxions of both expressions be taken, making $\dot{x}$ alone variable; whence, after dividing by $\ddot{x}$, we have $\frac{3y\dot{x}\ddot{x}}{\dot{y}}$ and 1 : therefore, in this case $\frac{3y\dot{x}\ddot{x}}{\dot{y}} + e = 0$: whence $\dot{x} = a^{\frac{1}{3}} y^{-\frac{1}{3}} \dot{y}$ (supposing $a = -\frac{1}{3}e$); and consequently $x = 2a^{\frac{1}{3}} y^{\frac{2}{3}}$; being an equation answering to the *common parabola*. The same conclusion may be otherwise derived (without second-fluxions) by assuming $\frac{\dot{x}}{\dot{y}} = v$; whereby our two given expressions will be transformed to $y\dot{y}v^2$ and $\dot{y}v$: from whence, by the rule, we get $3v^2y\dot{y} + e\dot{y} = 0$; and therefore $v \left(\frac{\dot{x}}{\dot{y}}\right) = a^{\frac{1}{3}} y^{-\frac{1}{3}}$; whence $\dot{x} = a^{\frac{1}{3}} y^{-\frac{1}{3}} \dot{y}$, and consequently $x = 2a^{\frac{1}{3}} y^{\frac{2}{3}}$, the same as before.

Fig. 23. If the *abscissa* (AP) of a curve AQC be denoted by x , and the *ordinate* PQ by y , and p be taken to express the measure of the circumference of a circle whose diameter is unity; it is well known that the several fluxions, of the *abscissa* AP , curve-line AQ , area APQ , superficies of the generated solid (by a rotation about the axis AP), and of the solid itself, will be, respectively, represented by $\dot{x}$, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, $y\dot{x}$, $2py\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and $py^2\dot{x}$: if therefore, the fluxions of these different expressions be taken, as before (making $\dot{x}$ alone variable) we shall get $1 + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + f\dot{y} + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + h\dot{y}^2 = 0$; being a general equation for determining the relation of x and y , when any one of those five quantities (*viz.* the *abscissa*, curve-line, area, superficies, or solid) is a *maximum* or *minimum*, and all,

or any number of the others, at the same time, equal to given quantities; wherein the coefficients e , f , g , and h , may be positive, negative, or nothing, as the case proposed may require. Thus, for instance, if the length of the curve only be given, and the area corresponding is required to be a *maximum*, our equation will then become $\frac{ex}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + fy = 0$, or $a^2 \dot{x}^2 = y^2 \times \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ (by making $a = -\frac{e}{f}$); whence $\dot{x} = \frac{y\dot{y}}{\sqrt{aa - yy}}$, and consequently $x = a - \sqrt{aa - yy}$, or $2ax - x^2 = y^2$; answering to a circle; which figure is, therefore, more capacious than any other under equal bounds.

If, together with the ordinate (which, here, is always supposed given) the abscissa, at the end of the fluent, be given likewise, and the superficies generated by the rotation of the curve about its axis be a *minimum*; then, from the same equation, we shall have $1 + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = 0$, whence (making $a = -\frac{1}{g}$) $\dot{x}$ is found $= \frac{ay}{\sqrt{yy - aa}}$, and from thence $x = a \times \text{hyp. log. } \frac{y + \sqrt{yy - aa}}{a}$: which equation, by being impossible when y is less than a , shews that the curve (which is here the *catenaria*) cannot possibly meet the axis about which the solid is generated; and consequently, that the case will not admit of any *minimum*, unless the first, or least given value of y , exceeds a certain assignable magnitude.

When any, or all of the above-specified quantities are given, and the contemporary fluent of some other expression, as $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}^m \times y^n \dot{y}^{1-2n}$, is required to be a *maximum* or *minimum*, our equation (by taking the fluxion of this last expression, and joining it to the former) will then be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}^{m-1} \times 2nxy^{1-2n} + d + \frac{ex}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + fy + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + by^2 = 0$: which, when $m=1$, and $n=-1$, will be that defining the solid of the least resistance; and this, when the length of the axis, only, is supposed to be given (without farther restrictions) will be expressed

pressed by $\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-2} \dot{x} - 2\dot{x}\dot{y}\dot{y}' + d = 0$, or $2\dot{y}\dot{y}'\dot{x} = d \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}$; being the case first considered by Sir ISAAC NEWTON.—If both the length and the solid content be given, the equation will be $-2\dot{x}\dot{y}\dot{y}' \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-2} + d + b\dot{y}^2 = 0$; but if, besides these, the superficies is given likewise, it will then be $-2\dot{x}\dot{y}\dot{y}' \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-2} + d + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + b\dot{y}^2 = 0$.

Thus, in like manner, by assuming $m = -\frac{1}{2}$, and $n = \frac{1}{2}$, we have $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + d + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + f\dot{y} + \frac{gy\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + b\dot{y}^2 = 0$; being the general equation of the *curve of the swiftest descent*; which, when e , f , g , and b are all of them taken equal to nothing, will become $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + d$; which is the case considered by many Others, answering to the cycloid. When the length of the arch described in the whole descent (along with the values of x and y) is given, the equation will then be $\frac{y^{-\frac{1}{2}}\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} + d + \frac{e\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = 0$, or $e + y^{-\frac{1}{2}}\dot{x}^2 = d \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}$. And thus may the relation of x and y be determined in any other case, and under any number of restrictions; provided that one of these quantities, only, enters into the several expressions given.—When both x and y are concerned, as well as their fluxions, the consideration becomes more complicated; nor does it seem practicable to arrive at a General Rule, to answer equally in such cases. Nevertheless, if the ultimate values of x and y are supposed given, or the required curve is to pass through two given points, without being confined to farther limitations, except that of the maximum or minimum (which case is the chief, and the most useful that can occur); then the method of solution may be as follows:

Take the fluxion of the given expression (whose fluent is to be a maximum or minimum) making $\dot{x}$ alone variable; and, having divided by $\dot{x}$, let the quotient be denoted by u .

Take, again, the fluxion of the same expression, making x , alone, variable; which divide by $\dot{x}$: then will this last quotient $= u$.

From

From which equation the value of u , and the relation of x and y will be determined.

Thus, for example, if the expression propounded (whose fluent, corresponding to any given values of x and y , is to be a *minimum*) were to be $\sqrt{f+gx} \times \frac{y\dot{x}^3}{\dot{y}}$; then the fluxion thereof, when $\dot{x}$ alone is made variable, being $\sqrt{f+gx} \times \frac{3y\dot{x}^2\dot{x}}{\dot{y}}$, and, when x alone is made variable, equal to $\frac{gy\dot{x}^3}{\dot{y}}$, we here have $\sqrt{f+gx} \times \frac{3y\dot{x}^2}{\dot{y}} = u$, and $\frac{gy\dot{x}^3}{\dot{y}} = \dot{u}$; the latter of which, divided by the former, gives $\frac{\dot{u}}{u} = \frac{\frac{3g\dot{x}}{f+gx}}{\frac{3y\dot{x}^2}{\dot{y}}}$; whence hyp. log. $u = \text{hyp. log. } \sqrt{f+gx}^{\frac{1}{2}} + \text{hyp. log. } d$ (d being any constant quantity). Consequently $u = d \times \sqrt{f+gx}^{\frac{1}{2}}$; which value being substituted in the equation $\sqrt{f+gx} \times \frac{3y\dot{x}^2}{\dot{y}} = u$, we thence have $\sqrt{f+gx}^{\frac{1}{2}} \times \dot{x}^2 = \frac{1}{3} dy^{-1}\dot{y}^2$, or $\sqrt{f+gx}^{\frac{1}{2}} \times \dot{x} = cy^{-1}\dot{y}$ (making $c = \sqrt{\frac{d}{3}}$), and consequently by taking the fluent again, we have $\frac{3 \times \sqrt{f+gx}^{\frac{1}{2}} - 3f^{\frac{1}{2}}}{4g} = 2cy^{\frac{1}{2}}$; expressing the general relation of x and y , supposing them both to begin to be generated together.—If $f = 1$ and $g = 0$, the fluxion propounded will become $\frac{y\dot{x}^3}{\dot{y}}$ (the same as in the first of the former examples); and here, $\dot{x}$ being $= cy^{-1}\dot{y}$, x will be $= 2cy^{\frac{1}{2}}$, answering (as before) to the common *parabola*.—But if $f = 0$ and $g = 1$, then our given fluxion will become $\frac{xy\dot{x}^3}{\dot{y}}$, and the resulting equation will be $\frac{3}{4}x^{\frac{4}{3}} = 2cy^{\frac{1}{2}}$, or $x^3 = a'y^3$ (a being put $= \frac{8c}{3}$); which also answers to a *parabola*, but of an higher order.—The very same conclusions will, in like manner, be brought out by making $\dot{y}$ and y , successively, variable (instead of $\dot{x}$ and x). For, here, the two fluxions resulting (after having divided by $\dot{y}$

and y) appear to be $\frac{f+gx \times -2yx^3}{y^3} = u$, and $\frac{f+gx \times x^3}{y^2} = \dot{u}$:

whence, dividing the latter by the former, we have $\frac{-\frac{1}{2}\dot{y}}{y} = \frac{\dot{u}}{u}$;

and therefore — hyp. log. $y^{\frac{1}{2}} + \text{hyp. log. } a = \text{hyp. log. } u$
(a being any constant quantity). Consequently, $ay^{-\frac{1}{2}} = u =$

$\frac{f+gx \times -2yx^3}{y^3}$, and $f+gx \times x^3 = cy^{-\frac{1}{2}}y$ (c^3 being put =
— $\frac{1}{2}a$). Hence, by taking the fluent again, we have

$\frac{3 \times \frac{f+gx}{y^3} - 3f^{\frac{4}{3}}}{4g} = 2cy^{\frac{1}{2}}$, the very same as before.



Of the REDUCTION of Algebraic Equations, by the
Method of SURD DIVISORS; containing an Ex-
planation of the Grounds of that Method, as it
is laid down by Sir ISAAC NEWTON in his Uni-
versal Arithmetick.

***THE reduction of equations by surd divisors, which is
* T * looked upon, by many, as a very intricate kind of
*** speculation, is founded on the same principles with
the method of extracting the roots of common qua-
dratic equations, by compleating of the square, with this diffe-
rence only, that the squares on both sides of the equation are,
here, affected by the unknown quantity; whereas, in the com-
mon method, the square on the right-hand side is a quantity
intirely known. What we, therefore, have to do, is, *to*
separate, and so order the terms of the equation given, that both
sides thereof may (if possible) be complete squares.

CASE I. *If the given equation be a biquadratic one, let it be*
 $x^4 + px^3 + qx^2 + rx + s = 0$, and let there be assumed
 $xx + \frac{1}{2}px + \mathcal{Q}$ — $Ax + B$ = $x^4 + px^3 + qx^2 + rx + s (=0)$;
that is, let the values of the quantities $\mathcal{Q}$, A and B be supposed
such, that the coefficients of the powers of x , when $xx + \frac{1}{2}px + \mathcal{Q}$
and $Ax + B$ are squared, shall agree, or be the same, in every
term; with those of the equation given. Then, the said quan-
tities being actually squared, our equation will become

$$\left. \begin{array}{l} x^4 + px^3 + 2\mathcal{Q}x^2 \quad * \quad * \\ + \frac{1}{4}p^2x^2 + p\mathcal{Q}x + \mathcal{Q}^2 \\ - A^2x^2 - 2ABx - B^2 \end{array} \right\} = x^4 + px^3 + qx^2 + rx + s.$$

From whence, by equating the coefficients of the homologous
powers, and putting $a = q - \frac{1}{4}p^2$, we have,

1. $2\mathcal{Q} + \frac{1}{4}p^2 - A^2 = q$, or, $2\mathcal{Q} = A^2 + a$;
2. $p\mathcal{Q} - 2AB = r$, or, $p\mathcal{Q} = 2AB + r$;
3. $\mathcal{Q}^2 - B^2 = s$, or, $\mathcal{Q}^2 = B^2 + s$.

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Now, if the value of $\mathcal{Q}$, as given by the first equation, be substituted in the other two, we shall get $\frac{1}{2}pA^2 - 2AB = \beta$, and $\frac{1}{4}A^4 + \frac{1}{2}\alpha A^2 - B^2 = \zeta$, supposing $\beta = r - \frac{1}{2}\alpha p$, and $\zeta = s - \frac{1}{4}\alpha\alpha$. In which equations the unknown quantities appertaining to the latter of the two assumed squares are only concerned, and from which their values might be found. But as the resulting equation, when one of the quantities is exterminated, rises to the sixth dimension, and would, perhaps, require more trouble to reduce it than, even, the original one propounded, little advantage would be reaped therefrom. Instead, therefore, of proceeding farther in a direct manner, it may be of use to try, whether some property, or relation of these quantities cannot from hence be discovered, whereby we may be enabled to guess at their values; which may be afterwards tried by means of the equations here exhibited.

First, then, it is evident, if both A and B are either integers or rational quantities, that the equation $\overline{x^2 + \frac{1}{2}px + Q} - \overline{Ax + B}^2 (= x^2 + px^2 + qx^2 + rx) = 0$ will, even after it is reduced to $x^2 + \frac{1}{2}px + Q = Ax + B$, be intirely free from radical quantities. In which case, the method of *rational divisors* taking place, a reduction by means of surd quantities, or divisors, as they do not naturally arise in the consideration, cannot be of use. But the relation of the given quantities p, q, r, s (which we shall always, hereafter, consider as integers) may be such, that the values of A and B shall be radical quantities, commensurate to each other; in which case, where the method of *rational divisors* fails, we may assume $\sqrt{n}$ for the common radical divisor, and express the quantities themselves by $k\sqrt{n}$, and $l\sqrt{n}$; that is, we may make $A = k\sqrt{n}$, and $B = l\sqrt{n}$; by which means our two equations, derived above, will be changed to $\frac{1}{2}pk^2n - 2kl = \beta$, and $\frac{1}{4}k^4n + \frac{1}{2}\alpha k^2n - l^2 = \zeta$, or to $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, and $\frac{1}{4}k^4 + \frac{1}{2}\alpha k^2 - 2l^2 = \frac{2\zeta}{n}$, respectively.

Now, since n is supposed to be an integer, it is plain from hence (considering k and l also as integers, or the halves of such

such) that $\frac{\beta}{n}$ and $\frac{2\zeta}{n}$ must be integers likewise, or, at least, the halves of integers; and consequently that n (whose value we are here seeking) ought to be some common integral divisor of β and 2ζ .

Moreover, with regard to k and l , it is evident from the first of those equations ($\frac{1}{2}pk - 2l = \frac{\beta}{n}$) that the former (k) ought to be some divisor of $\frac{\beta}{n}$; and that, if the quotient $\frac{\beta}{n}$ be taken from $\frac{1}{2}pk$, the remainder ($\frac{1}{2}pk - \frac{\beta}{n}$) will be the double of l .

It farther appears, from the equations $\mathcal{Q} = \frac{A^2 + \alpha}{2}$, and $\mathcal{Q}^2 = B^2 + s$, by substituting for A^2 and B^2 their equals nk^2 and nl^2 , that $\mathcal{Q}$ will be $= \frac{\alpha + nk^2}{2}$, and $l^2 = \frac{\mathcal{Q}\mathcal{Q} - s}{n}$. From the former of which $\mathcal{Q}$ will be known, when n and k are known; by means whereof and the other equation, l may be, a second time, found; and the agreement, or coincidence of this value with that before determined for l , will prove the solution in all respects; because then the conditions of three original equations ($2\mathcal{Q} = A^2 + \alpha$, $p\mathcal{Q} = 2AB + r$, $\mathcal{Q}^2 = B^2 + s$) will be all completely fulfilled.—It is true indeed, that no immediate regard, in the conclusion, is had to the second of those equations; but then it ought to be observed, that the equation $\frac{1}{2}pk - \frac{\beta}{n}$, whereby l is, the first time, found, is a consequence thereof, being derived from *that*, and the first equation, conjunctly: and it is known, that, whatever values are discovered for unknown quantities, by means of equations derived from others, such values do equally answer the conditions of the original equations propounded.

Seeing the method of solution, above traced out, depends upon the assuming proper divisors of β , 2ζ , and $\frac{\beta}{n}$, for the values

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values of n and k , it may therefore be expedient, first of all, in order to bring the work into less compass, to reject such divisors of those quantities (if we can by any means discover them) which we know are not for our purpose. And this may, in some measure, be effected, from the consideration of the properties of even and odd numbers.

In order to which $Q = \frac{a + nk^2}{2}$ being previously transformed to $nk^2 (= 2Q - a) = 2Q - q + \frac{1}{2}p^2 = \frac{1}{2}p^2 - f$ (by putting $q = 2Q - f$), it is evident, from thence, that if p be an odd number, $p^2 = 4f$, and consequently its equal $4nk^2$, will likewise be an odd number; because an even number ($4f$) subtracted from the square of an odd one, always, leaves odd. Therefore, seeing $4k^2 \times n$ is here an odd number, both n and $4k^2$ must be odd (for the product of two even numbers, or of an odd one and an even one, is even, and not odd). Whence it follows, because $\frac{1}{2}p^2$ is odd, that $2k$ must be odd too; and consequently k the half of an odd number.

Now, seeing p , n , and $2k$ are all of them odd numbers (when p is such) they may, therefore, be expressed by $2a + 1$, $2b + 1$, and $2c + 1$, respectively; a , b , and c being integers: in consequence of which assumption the equation $4nk^2 = p^2 - 4f$, will, by substitution, be changed to $8bc^2 + 8bc + 2b + 4c^2 + 4c + 1 = 4a^2 + 4a + 1 - 4f$, or $2bc^2 + 2bc + \frac{1}{2}b + c^2 + c = a^2 + a - f$. From whence it is manifest, as all the terms, but $\frac{1}{2}b$, are known to be integers, that $\frac{1}{2}b$ must be an integer likewise: and so, b being an even number, it follows that n , or $2b + 1$, must be the double of an even number (or a multiple of 4) increased by unity. Therefore all the divisors of β and 2ζ that have not this property may be safely rejected, as not for the purpose.

In like manner, if p be even, the same limitations will take place, provided that r is odd; which will be the case when Q is the half of an odd number (For, when Q is an integer, $A^2 = \frac{1}{2}p^2 - f$) and $B^2 (= Q^2 - r)$ being integers, their product A^2B^2 will be an integer, and consequently the square root thereof AB (being rational) will likewise be an integer; and

and so, pQ and $2AB$ being both even numbers, their difference r , as given by the equation $pQ = 2AB + r$, would be even, and not odd). Therefore, seeing B^2 , or its equal nl^2 , is here equal to the square of half of an odd number (Q) joined to an integer ($-s$), in the same manner as nk^2 was in the preceding case; it is evident, from the reasoning there laid down, that the value of n is subject to the very same restrictions here, as there.—Other limitations might be pointed out, from the properties of even and odd numbers, were the thing worth pursuing farther. What is already delivered on this head is sufficient for the purpose, and for the understanding of Sir ISAAC NEWTON: I shall therefore, from the several conclusions above derived, now lay down the subsequent

R U L E

for the reduction of an equation ($x^4 + px^3 + qx^2 + rx + s = 0$) of four dimensions.

Make $\alpha = q - \frac{1}{4}p^2$, $\beta = r - \frac{1}{2}p\alpha$, and $\zeta = s - \frac{1}{4}\alpha\alpha$; then put for n some common integral divisor of β and 2ζ , that is neither a square, nor divisible by a square, and which being divided by 4, shall leave unity, if either p or r be odd. Put also for k some divisor of $\frac{\beta}{n}$, if p be even, or half of the odd divisor if p be odd: take the quotient from $\frac{1}{2}pk$, and call half the remainder l . Make $\mathcal{Q} = \frac{\alpha + nkk}{2}$, and try if n divides $\mathcal{Q}\mathcal{Q} - s$, and the root of the quotient be equal to l ; if it so happen, then the proposed equation, by means of the values thus determined, will be reduced to $xx + \frac{1}{2}px + \mathcal{Q} = \pm \sqrt{n} \times kx + l$.

That the divisor n ought not here to be a square, is evident from what has been already remarked, since both A and B would then be rational quantities; and that it ought not to be divisible by a square, will also appear, if it be considered that k and l in the equations $k\sqrt{n} = A$, and $l\sqrt{n} = B$, are to be taken the greatest, and n the least, that the case will admit of.

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No regard in this Rule is had to that circumstance, in which β happens to be nothing. Sir ISAAC NEWTON here directs, *to take k also equal to nothing*. The reason of which depends on the equation $\frac{1}{2}pk^2 - 2kl = \frac{\beta}{n}$, which in this case becomes $\frac{1}{2}pk^2 - 2kl = 0$; where one root, or value of k must, necessarily, be nothing. Therefore Q being $= \frac{1}{2}\alpha$, we have $\sqrt[n]{n} (= \sqrt{Q^2 - s}) = \sqrt{\frac{1}{4}\alpha^2 - s}$; so that, by a direct process, our given equation is here reduced to $x^2 + \frac{1}{2}px + \frac{1}{2}\alpha = \sqrt{\frac{1}{4}\alpha^2 - s}$, wherein α is given $= q - \frac{1}{2}pp$.

The celebrated mathematician MACLAURIN, who, in his *Treatise of Algebra*, has commented largely on the discoveries of our Author, seems to represent this part of the General Rule, as not well grounded; laying down, at the same time, two new Rules, in order to supply the defect. Which Rules, I must confess, to me appear unnecessary; since it is certain, that the method of solution, as laid down by Sir ISAAC NEWTON, is more direct and eligible in this particular case than in any other. It must be allowed, indeed, that the manner of applying the Rule, in this case, is left somewhat obscure; but as to his directing, to take $k = 0$, when $\beta = 0$, it cannot, I am fully persuaded, admit of any well-grounded objection. For, though "it does not necessarily follow that k must be $= 0$, when $\beta = 0$," yet the taking of k thus $= 0$, involves no absurdity; seeing one value of k (at least) will be nothing.—The truth is, there are three different values that k may admit of (as appears by the subsequent note*); all of which will, equally, fulfil the several conditions required, and bring out the very same conclusion. Thus the value of k , in the equation

* If the square of half the second of the original equations, $2Q - \alpha = AA$, $pQ - r = 2AB$, $QQ - s = BB$, be subtracted from the product of the other two, there will be obtained the equation $Q^3 - \frac{1}{2}qQ^2 + \frac{1}{4}pr - s \times Q + \frac{1}{2} \times \alpha s - \frac{1}{4}rr = 0$; wherein the unknown quantity Q is alone concerned; which equation being of three dimensions, the root Q , and consequently k ($\frac{2Q - \alpha}{n}$) will admit of three different values.—From this equation it also appears, that Q must always be a divisor of the quantity $\alpha s - \frac{1}{4}rr$; which is a circumstance taken notice of by our Author.

$x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$ (proposed by this gentleman) may be 0, 3 or 4; or, which comes to the same, the equation itself

may be reduced to $x^2 + x - 19 = \pm 6\sqrt{10}$, $x^2 + x + \frac{7}{2}$

$= \pm \sqrt{5 \times 3x + \frac{3}{2}}$, or to $x^2 + x - 3 = \pm \sqrt{2 \times 4x + 2}$.

All which are, in effect, but one and the same equation, as will appear by squaring both sides of each, and properly transposing; from whence the given equation $x^4 + 2x^3 - 37x^2 - 38x + 1 = 0$, will in every case emerge. The second of these equations is that brought out by Mr. MACLAURIN; but the first, which is that found by our Author's Rule, is not only more commodious, but easier to be determined, being derived by a direct, and very short process.—And so much for equations of four dimensions.

CASE. II. If the equation to be reduced is of six dimensions, let it be $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$; and let there be assumed $\frac{x^3 + \frac{1}{2}px^2 + Qx + R}{Ax^2 + Bx + C} = \frac{x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v}{Ax^2 + Bx + C} (= 0)$; which, by involution and transposition, will give

$$\left. \begin{aligned} &+ 2Qx^4 + 2Rx^3 \\ &+ \frac{1}{4}p^2x^4 + pQx^3 + pRx^2 \\ &+ Q^2x^2 + 2QRx + R^2 \end{aligned} \right\} = \begin{aligned} &A^2x^4 + 2ABx^3 + \\ &2ACx^2 + B^2x^2 + \\ &2BCx + C^2. \end{aligned}$$

$$- qx^4 - rx^3 - sx^2 - tx - v$$

From whence, by equating the coefficients of the homologous powers, and writing $\alpha = q - \frac{1}{2}pp$, we have,

1. $2Q - \alpha = A^2$;
2. $2R + pQ - r = 2AB$;
3. $pR + Q^2 - s = 2AC + B^2$;
4. $2QR - t = 2BC$;
5. $R^2 - v = C^2$.

If now the value of $Q (= \frac{1}{2}A^2 + \frac{1}{2}\alpha)$ as given by the first of these equations, be substituted for Q , in the second, we shall get $2R + \frac{1}{2}pA^2 + \frac{1}{2}p\alpha - r = 2AB$; and consequently $R = AB - \frac{1}{4}pA^2 + \frac{1}{2}\beta$ (by putting $\beta = r - \frac{1}{2}p\alpha$): which value, together with that of Q , being substituted in the three remaining equations, we shall have,

Q

I.

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$$\begin{aligned} 1. pAB - \frac{1}{4}p^2A^2 + \frac{1}{2}p\beta + \frac{1}{4}A^4 + \frac{1}{2}\alpha A^2 + \frac{1}{4}\alpha^2 - s &= 2AC + B^2, \\ 2. A^3B - \frac{1}{4}pA^4 + \frac{1}{2}\beta A^2 + \alpha AB - \frac{1}{4}p\alpha A^2 + \frac{1}{2}\alpha\beta - t &= 2BC, \\ 3. A^2B^2 - \frac{1}{2}pA^3B + \beta AB + \frac{1}{16}p^2A^4 - \frac{1}{4}p\beta A^2 + \frac{1}{4}\beta^2 - v &= C^2: \end{aligned}$$

which, by putting $\gamma = s - \frac{1}{2}p\beta$, $\zeta = \gamma - \frac{1}{4}\alpha\alpha$, $\eta = t - \frac{1}{2}\alpha\beta$, and $\theta = v - \frac{1}{4}\beta\beta$, will be reduced to

$$\begin{aligned} pAB - \frac{1}{4}p^2A^2 + \frac{1}{4}A^4 + \frac{1}{2}\alpha A^2 - \zeta &= 2AC + B^2, \\ A^3B - \frac{1}{4}pA^4 + \frac{1}{2}\beta A^2 + \alpha AB - \frac{1}{4}p\alpha A^2 - \eta &= 2BC, \text{ and} \\ A^2B^2 - \frac{1}{2}pA^3B + \beta AB + \frac{1}{16}p^2A^4 - \frac{1}{4}p\beta A^2 - \theta &= C^2, \text{ respectively.} \end{aligned}$$

Now, if the values of A , B and C are such, as to admit of some common surd-divisor, let that divisor (*as in the preceding Case*) be denoted by $\sqrt{n}$, and the quantities themselves by $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, respectively: then, substitution being made and every equation divided by n , we shall have,

$$\begin{aligned} 1. pkl - \frac{1}{4}p^2k^2 + \frac{1}{4}nk^4 + \frac{1}{2}\alpha k^2 - \frac{\zeta}{n} &= 2km + l^2, \\ 2. nk^3l - \frac{1}{2}pnk^4 + \frac{1}{2}\beta k^2 + \alpha kl - \frac{1}{4}p\alpha k^2 - \frac{\eta}{n} &= 2lm, \\ 3. nk^3l^2 - \frac{1}{2}pnk^3l + \beta kl + \frac{1}{16}p^2nk^4 - \frac{1}{4}p\beta k^2 - \frac{\theta}{n} &= m^2. \end{aligned}$$

From whence it appears (since k , l , and m are here considered as integers, or as the halves of such) that ζ , η , and θ ought to be all of them divisible by n ; or, which is the same, that n ought to be some common integral divisor of the quantities ζ , η , and θ *.

Furthermore, with respect to the limitations to which k is subject, let the several terms in the former part of the first of our three equations, in which k is found (in order to abbreviate the work) be denoted by Fk ; then will the equation itself be changed to $Fk - \frac{\zeta}{n} = 2km + l^2$. And, in the very same

* Sir ISAAC NEWTON directs to take n , some common divisor of 2ζ , η , and 2θ (instead of ζ , η , and θ); but this makes no difference, because all divisors of ζ and θ are also divisors of 2ζ and 2θ ; nor are there any divisors of 2ζ and 2θ , but what will likewise be divisors of ζ and θ , if we (as Sir ISAAC has done) admit into the consideration such fractions as have the powers 2 for their denominator; which arise from the value of $\frac{1}{2}p$, in the assumed equation, being a fraction of this kind, when p is an odd number.

manner,

manner, our other two equations will be changed to $Gk - \frac{\eta}{n} = 2lm$, and $Hk - \frac{\theta}{n} = m^2$, respectively.

Let now the square of half the second of these equations be subtracted from the product of the first and third, then will

$$F H k^2 - \frac{F k \theta}{n} - \frac{H k \xi}{n} + \frac{\xi \theta}{nn} - \frac{1}{4} G^2 k^2 + \frac{G k \eta}{2n} - \frac{1}{4} \eta^2 = 2 k m^2 ;$$

which, by dividing the whole by $2k$, and putting $\lambda = \xi \theta - \frac{1}{4} \eta^2$, will, at length, become

$$\frac{1}{2} F H k - \frac{1}{8} G^2 k - \frac{1}{2} H \times \frac{\xi}{n} + \frac{1}{4} G \times \frac{\eta}{n} - \frac{1}{2} F \times \frac{\theta}{n} + \frac{\lambda}{2nn} = m^2 ;$$

where ξ , η , and θ being all divisible by their common divisor n (as is shewn above) it is manifest that $\frac{\lambda}{2nn}$ (in order that m may be an integer, or the half of an integer) ought also to be divisible by its divisor k , that is, k ought to be some divisor of the quantity $\frac{\lambda}{2nn}$.

Again, with regard to l , let the value of $R (= AB - \frac{1}{2} pQ + \frac{1}{2} r)$, as given by the second of the five original equations, be substituted in the fourth; by which means we have $2QAB = pQ^2 + rQ - t = 2BC$, or $2Qnkl - pQ^2 + rQ - t = 2nlm$ (because $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$), and consequently $\frac{rQ - pQ^2 - t}{nl} = 2m - 2kQ$; where $2m - 2kQ$ being an integer, it is evident that $\frac{rQ - pQ^2 - t}{nl}$ must be an integer also,

and, consequently, l some divisor of the quantity $\frac{rQ - pQ^2 - t}{n}$.

From whence l being found in numbers, the value of $R (= AB - \frac{1}{2} pQ + \frac{1}{2} r = nkl - \frac{1}{2} pQ + \frac{1}{2} r)$ will be had likewise; and then, by means of the three last of the five original equations, the value of m may be also found, three several ways, and the truth of the solution thereby confirmed: for these equations, by substituting for A , B , and C , their equals $k\sqrt{n}$, $l\sqrt{n}$, and $m\sqrt{n}$, do become $R^2 - v = nm^2$, $2QR - t = 2nlm$, and $pR + Q^2 - s = 2nkm + nll$; from the first of which

Q 2

m

$m = \sqrt{\frac{R^2 - v}{n}}$; from the second, $m = \frac{QR - \frac{1}{2}t}{nl}$; and from the third, $m = \frac{QQ + pR - nll - s}{2nk}$: which values, therefore, when Q, R, &c. are rightly assumed, will be all found equal among themselves; and our given equation, $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$, or $x^3 + \frac{1}{2}px^2 + \frac{Qx + R}{Ax^2 + Bx + C} = 0$, will then be reduced to $x^3 + \frac{1}{2}px^2 + Qx + R (= \pm \frac{Qx + R}{Ax^2 + Bx + C}) = \pm \sqrt{n \times kx^2 + lx + m}$.

As to the limitations in the divisors to be tried, with respect to even and odd numbers, the reasoning thereon is the very same as in the preceding Case; which, therefore, it will be unnecessary to repeat.—One circumstance there is, indeed, that merits a particular regard, and that is, when $\lambda = 0$; in which case k (or one value of k at least) will also be $= 0$, and the reduction will be performed by a direct process. For, k being nothing, the three equations wherein k , l , and m are first introduced, will become $-\frac{\zeta}{n} = l^2$, $-\frac{n}{n} = 2lm$, and $-\frac{\theta}{n} = m^2$; whence $l\sqrt{n} = \sqrt{-\zeta}$, $m\sqrt{n} = \sqrt{-\theta}$, and consequently $\lambda (= \zeta\theta - \frac{1}{4}\eta^2) = 0$, as it ought to be. Therefore, by substituting these values, and writing also instead of Q and R their equals $\frac{1}{2}\alpha$ and $\frac{1}{2}\beta$, the equation given, is here reduced to $x^3 + \frac{1}{2}px^2 + \frac{1}{2}\alpha x + \frac{1}{2}\beta = \pm x\sqrt{-\zeta} \pm \sqrt{-\theta}$.

CASE III. If the equation is of eight dimensions, let it be $x^8 + px^7 + qx^6 + rx^5 + sx^4 + tx^3 + vx^2 + wx + z = 0$: Then, by assuming $x^4 + \frac{1}{2}px^3 + \frac{Qx^2 + R}{Ax^2 + Bx + C} = x^4 + \frac{1}{2}px^3 + \frac{Qx^2 + R}{Ax^2 + Bx + C} = x^4 + \frac{1}{2}px^3 + \frac{Qx^2 + R}{Ax^2 + Bx + C}$, and proceeding as in the two former Cases, we shall have here,

1. $\zeta - \alpha \dots \dots \dots = A^2$,
2. $2R + pQ - r \dots \dots = 2AB$,
3. $2S + pR + QQ - s = 2AC + BB$,
4. $pS + 2QR - t \dots \dots = 2AD + 2BC$,
5. $2QS + RR - v \dots \dots = 2BD + CC$,

$$6. 2RS - w = 2CD,$$

$$7. SS - z = DD.$$

Put now (as before) $A = k\sqrt{n}$, $B = l\sqrt{n}$, $C = m\sqrt{n}$, $D = n'\sqrt{n}$; put also (to shorten the work) $Q = Q'n + \frac{1}{2}\alpha$, $R = R'n + \frac{1}{2}\beta$, $S = S'n + \frac{1}{2}\gamma$; that is, let the quotients of Q , R , and S , when divided by the common divisor n , be Q' , R' , and S' , and the remainders $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$, respectively: then, to determine these last, which must be first known, before n can be known, let substitution be made in the second and third equations, every-where disregarding such terms wherein n and its powers are involved. Thus, substitution being made in the second equation, we have $2kn + \beta + pQ'n + \frac{1}{2}p\alpha - r = 2kln$; where the homologous terms, in which n enters not, are β , $\frac{1}{2}p\alpha$, and $-r$: the others, therefore, being here disregarded, we have $\beta + \frac{1}{2}p\alpha - r = 0$, or $\beta = r - \frac{1}{2}p\alpha$. In the very same manner, from the third equation, $\gamma + \frac{1}{2}\beta + \frac{1}{4}\alpha\alpha - s = 0$, and consequently $\gamma = s - \frac{1}{2}\beta - \frac{1}{4}\alpha\alpha$.

Let substitution be now made in the fourth, fifth, sixth, and seventh equations (still disregarding all such terms as would involve the powers of n), and there will come out,

$$1. \frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta - t = ,$$

$$2. \frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta - v = ,$$

$$3. \frac{1}{2}\beta\gamma - w = ,$$

$$4. \frac{1}{4}\gamma\gamma - z = .$$

Now, as all the other terms, that would arise in these equations (besides those put down) are affected with n , and are therefore divisible thereby, it is manifest that the four quantities $\frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta - t$, $\frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta - v$, $\frac{1}{2}\beta\gamma - w$, and $\frac{1}{4}\gamma\gamma - z$, here brought out, must likewise be, all of them, divisible by the same common divisor n , when the equation given is capable of being reduced. If, therefore, no such common divisor (under the restrictions specified in the preceding Cases, depending on p , r , t , or w being an odd number) can be discovered (which will most commonly happen) the work will then be at an end.

From the same method of operation, which may be looked upon as a sort of examination, whether the equation be reducible

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cible or not, we may find all the quantities to which n ought to be a common divisor, when the equation given is of 10, 12, or a greater number of dimensions.

Thus, let there be given $x^{2e} + px^{2e-1} + qx^{2e-2} + rx^{2e-3} + sx^{2e-4} + tx^{2e-5} \mathcal{E}c. = 0$, and let there be assumed

$$x^e + \frac{1}{2}px^{e-1} + Q'n + \frac{1}{2}\alpha \times x^{e-2} + R'n + \frac{1}{2}\beta \times x^{e-3} + S'n + \frac{1}{2}\gamma \times x^{e-4} \mathcal{E}c. - n \times kx^{e-1} + lx^{e-2} + mx^{e-3} \mathcal{E}c.]^2 = x^{2e} + px^{2e-1} + qx^{2e-2} \mathcal{E}c.$$

then, by squaring $x^e + \frac{1}{2}px^{e-1} + Q'n + \frac{1}{2}\alpha \times x^{e-2} \mathcal{E}c.$ and transposing $x^{2e} + px^{2e-1} + qx^{2e-2} \mathcal{E}c.$ it will appear, that the terms of this equation, in which n enters not, will be

$$\left. \begin{array}{l} \alpha \\ -q \end{array} \right\} \times x^{2e-2} + \left. \begin{array}{l} \beta \\ -r \end{array} \right\} \times x^{2e-3} + \left. \begin{array}{l} \gamma \\ -s \end{array} \right\} \times x^{2e-4} + \left. \begin{array}{l} \delta \\ -t \end{array} \right\} \times x^{2e-5} + \left. \begin{array}{l} \epsilon \\ -v \end{array} \right\} \times x^{2e-6}$$

$\mathcal{E}c.$ From the former half of which terms, all the quantities $\alpha, \beta, \gamma, \mathcal{E}c.$ will be determined, by assuming the coefficients equal to nothing: thus we have $\alpha = q - \frac{1}{2}p\beta, \beta = r - \frac{1}{2}p\alpha, \gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha, \delta = t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta, \mathcal{E}c.$ And then, these quantities being known, the coefficients of the remaining terms will likewise be known; which ought, all of them, to be divisible by n , in order that the reduction may succeed; that is, they ought to be such, as to admit of a common divisor (n) under the restrictions before specified.

For example, if the equation given were to be of twelve dimensions, as $x^{12} + px^{11} + qx^{10} + rx^9 + sx^8 + tx^7 + vx^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, we should have $\alpha = q - \frac{1}{2}p\beta, \beta = r - \frac{1}{2}p\alpha, \gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha, \delta = t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta,$ and $\epsilon = v - \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\epsilon\beta$; and the coefficients of the other six terms (whereof n ought to be a common divisor) would be $\frac{1}{2}p\epsilon + \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma - a, \frac{1}{2}\alpha\epsilon + \frac{1}{2}\beta\delta + \frac{1}{4}\gamma\gamma - b, \frac{1}{2}\beta\epsilon + \frac{1}{2}\gamma\delta - c, \frac{1}{2}\gamma\epsilon + \frac{1}{2}\delta\delta - d, \frac{1}{2}\delta\epsilon - e,$ and $\frac{1}{4}\epsilon\epsilon - f.$

These operations, for finding of n , as this sort of reduction is seldom possible in high equations, will most commonly end the work. If such a value, however, should be found for n , as to answer all the conditions above specified, it is not by pursuing the same method of divisors, laid down in the resolution of

of the preceding Cases, that the values of k , l , &c. can from thence be determined, without a prodigious deal of trouble. There are indeed various other means of trying these quantities, by assuming some of them, and finding the others from thence; and so proceeding on, changing the values continually, till all the conditions of the several equations, arising from the comparison of the homologous terms, are fulfilled. But as this is exceedingly laborious, and seeing after all, the use of so great reductions (as the sagacious Author himself observes) is very little, there not being, perhaps, one case in a thousand in which they can succeed; I shall, therefore, desist here.





THE R E S O L U T I O N O F

Some GENERAL PROBLEMS in MECHANICS, and
PHYSICAL ASTRONOMY.

P R O B L E M I.

Fig. 24. Suppose a system of bodies A, B, C (connected together) to revolve about a center, or axis (P), with a given angular celerity; it is proposed to find the momentum (k) which, acting at a given distance QP from the center, shall be just sufficient to stop, or take away the whole motion of the system.



If the given angular celerity of the system, at any distance PG from the axis, be denoted by v , the celerities of the several bodies A, B, and C will be truly expressed by $\frac{AP}{PG} \times v$, $\frac{BP}{PG} \times v$, and $\frac{CP}{PG} \times v$, respectively. Hence (by the property of the *Lever*) it will be, as PQ is to AP, so is $(\frac{AP}{PG} \times v \times A)$ the momentum of the body A, to $\frac{AP^2}{QP \times PG} \times v \times A$, the momentum, which acting at Q, is a just counterpoise to the action of A. And, in the very same manner, the momentum, acting at Q, sufficient to take away the motion of B, appears to be $\frac{BP^2}{QP \times PG} \times v \times B$; and so on. Whence it is manifest, that the sum of all these, must be the true momentum required; or that $k = v \times \frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{PG \times QP}$.

Q. E. I.

C O R O L L A R Y I.

If the motion of the system is that which might be produced by any given momentums a , b , c (or forces capable of producing those momentums) acting on the bodies A, B, C, in directions

directions perpendicular to AP, BP, and CP; then (by the property of the lever) the force $a \times \frac{AP}{QP}$, acting at Q, having the same effect to turn the system about its axis, as the force a , acting at the distance AP, &c. it follows that the force, which, by acting at Q, is sufficient to destroy the whole motion of the system, will here be $a \times \frac{AP}{QP} + b \times \frac{BP}{QP} + c \times \frac{CP}{QP}$: which being substituted in the room of k , our general equation, in the last article, will become $v = PG \times \frac{a \times AP + b \times BP + c \times CP}{A \times AP^2 + B \times BP^2 + C \times CP^2}$; shewing the angular celerity at the distance PG, produced in the system by the action of the given forces; which celerity is, therefore, in proportion to the celerity $\left(\frac{a}{A}\right)$ that the given force (or momentum) a is capable of producing in the single body A, as $\frac{A \times PG}{a} \times \frac{a \times AP + b \times BP + c \times CP}{A \times AP^2 + B \times BP^2 + C \times CP^2}$ to unity.

COROLLARY II.

If the momentum k be given equal to *that* of the whole system (A, B, C) in a direction perpendicular to the line PGQ passing through the common center of gravity G; then the length of the lever (PQ) by which k acts, may be determined from hence. For, the celerity of the point G being represented by v , the momentum last named will, by the property of the center of gravity, be rightly defined by $v \times A + B + C$; which being substituted in the room of k , we thence get $QP = \frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A + B + C \times GP}$; exhibiting the distance of the *center of percussion* (Q) at which an immovable obstacle receives the whole force of the stroke.

COROLLARY III.

If a single body S, equal to the sum of all the bodies A, B, C, be supposed to revolve (independent of the others) about the same center, with the common angular celerity of

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The Resolution of some General Problems

the system, its momentum $\frac{SP}{PG} \times v \times S$, or $\frac{SP}{PG} \times v \times \overline{A+B+C}$, will be in proportion to the momentum k (given by the Proposition) as $QP \times SP$ to $\frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A+B+C}$. By making these two quantities equal to each other, we have $SP = \frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A+B+C \times QP}$, for the distance of the body S from the axis of motion, when its momentum is equal to the momentum k , or when equal forces, applied to the single body at S , and to the system at Q , can take away, or produce equal angular celerities in both, about the common axis of motion P .

COROLLARY IV.

Hence, if the point Q be supposed to coincide with S , our last equation will become $SP = \sqrt{\frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A+B+C}}$, shewing the distance of the *center of gyration*, or the place of the body S , where the same force can take away, or produce the whole motion of the system A, B, C , as can take away, or produce the motion of the single body S , equal to the sum of all the former, and revolving with the same angular celerity.

COROLLARY V.

But if the point Q be taken in the common center of gravity G of the system, and Gg and Ss be drawn perpendicular to the horizontal line TP ; then, the force of gravity by which the whole system is urged in the direction Gg perpendicular to the horizon, being the sum of all the weights $(A+B+C)$ it is plain that the part of it acting in a direction perpendicular to PS , whereby the motion about the center is accelerated, will be $\overline{A+B+C} \times \frac{Pg}{PG}$. But the force whereby the weight S , in a direction perpendicular to the same PS , is accelerated, is equal to $S \times \frac{Ps}{PS} = \overline{A+B+C} \times \frac{Pg}{PG}$ (because $S = A+B+C$, and $\frac{Ps}{PS} = \frac{Pg}{PG}$). Therefore, seeing the forces acting at S
and

and Q are here equal, it is evident, from *Corol. III.* that the distance SP , so that the same angular celerity may be produced in the single body as in the system, will be truly exhibited by the general equation $SP = \frac{A \times AP^2 + B \times BP^2 + C \times CP^2}{A + B + C \times QP}$ there derived; the point S thus determined being the center of oscillation, and the same with the center of percussion, found in *Cor. II.* having its distance from the axis, equal to a third proportional to the distance of the center of gravity and that of gyration, determined in *Corol. IV.*

COROLLARY VI.

Hence, also, the pressure on the axis of suspension P may be deduced: for, since the angular celerities, produced in the system, and in the single body S , by the equal forces $\frac{A + B + C}{A + B + C} \times \frac{Pg}{PG}$ and $S \times \frac{Ps}{PS}$, are the same, it is manifest that the absolute celerity produced in G , during any given time, will be but the $\frac{PG}{PS}$ part of that produced in S ; so that only the $\frac{PG}{PS}$ part of the gravity of the system is employed in accelerating its motion, the other part $\frac{GS}{PS}$ being lost on the axis of suspension; which axis will therefore, in a direction perpendicular to PG , sustain a force expressed by $\frac{A + B + C}{A + B + C} \times \frac{Pg}{PG} \times \frac{GS}{PS}$. But this is not the only force by which the axis is affected; since, besides the other part of the force of gravity $(\frac{A + B + C}{A + B + C} \times \frac{Gg}{PG})$ in the direction GP , the centrifugal force, acting in the same direction, is to be taken into the consideration; whereof the quantity will be the same, as if the whole mass of the system was to be placed in its common center of gravity G . For, if upon PS the perpendiculars Aa , Bb , Cc be let fall; then, the centrifugal forces of the several bodies A , B , C being as the masses drawn into the respective distances from the center P , the effect of those forces in the direction PG ,

Fig. 25.

will therefore be expressed by $A \times Pa + B \times Pb + C \times Pc$, which (by the property of the center of gravity) is known to be equal to $A + B + C \times PG$.

But the pressure on the axis may be otherwise deduced, independent of the center of oscillation: for the angular celerity generated in the system about its center of gravity G (which is the same with the angular celerity about the point of suspension P) is intirely the effect of the action on the point of suspension; and the momentum, or force, sufficient to produce that celerity, is found (by the Proposition) to be

$$v \times \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{PG^2}; \text{ which is to the absolute mo-}$$

mentum $v \times \frac{A + B + C}{A \times AG^2 + B \times BG^2 + C \times CG^2}$, generated in the system, as to PG . Therefore the force act-

ing on the axis of suspension, in a direction perpendicular to PG , must be to the force employed in accelerating the motion of the system (in the like direction), in the same proportion above specified; so that, to have the true measure of each, the force of gravity must be divided in that ratio: whence (taking $GS = \frac{A \times AG^2 + B \times BG^2 + C \times CG^2}{A + B + C \times PG}$) it will be, as $GS + PG$

(PS) is to GS , so is the force of gravity, in a direction perpendicular to PG , to the force acting on the axis of suspension, in the like direction.

That the proportion here determined is the same with that found above, and the point S , the center of oscillation, is thus made to appear.

$$\text{Since } AP^2 = GP^2 + GA^2 - 2GP \times Ga,$$

$$BP^2 = GP^2 + GB^2 + 2GP \times Gb,$$

$$CP^2 = GP^2 + GC^2 - 2GP \times Gc,$$

it is evident that $A \times AP^2 + B \times BP^2 + C \times CP^2$ (as given above) will be $= \frac{A + B + C \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2 - 2GP \times A \times Ga - B \times Gb + C \times Gc}{A + B + C \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2}$, barely; because (from the property of the center of gravity) all the quantities $A \times Ga - B \times Gb + C \times Gc$ destroy one another. Hence,
by

by substituting the quantity here found, instead of its equal (*in Cor.*

$$V.) \text{ we get } SP = \frac{A + B + C \times GP^2 + A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C \times GP} \\ = GP + \frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C \times GP}; \text{ and consequently}$$

(SG) the distance of the center of oscillation, or percussion from the center of gravity = $\frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C \times GP}$, the

very same as above. Hence it also appears, that, if the plane of the motion remains unchanged, the rectangle under SG and GP will be a constant quantity; and that, if S be made the point of suspension, then P will become the center of oscillation; and, lastly, that the oscillations will be performed in the shortest time possible, when SG and GP are equal to one another, and equal, each, to $\sqrt{\frac{A \times GA^2 + B \times GB^2 + C \times GC^2}{A + B + C}}$; it being well known, that the sum of two lines, whose rectangle is given, will be a *minimum* when the lines themselves are equal to each other.

The same method laid down above, for finding the pressure upon the axis of suspension at rest, answers equally when that axis is supposed to have a motion, or when the system, or body, has a progressive motion, as well as an angular one (as is the case of a cylinder, which, in its descent, is made to revolve about its axis, by means of a rope wrapped about it, whereof one end is made fast at the place from whence the motion commences): the momentum of the rotation about the center of gravity, generated in a given particle of time, being always as the force producing it, drawn into the distance of the point where the force acts, from the center of gravity, as well when that point is in motion, as when it is at rest.

Another thing it may be proper to take notice of, which is, that in the foregoing considerations the bodies A, B, C are supposed to be very small; so as to have all their parts, nearly, at the same distance from the axis of motion. But, to have the conclusion accurately true, every particle of matter in the system ought

ought to be considered, and treated, as a distinct body: from whence, by means of the *method of fluxions*, the sum of all the *momenta* will be truly found: but this relating merely to matters of calculation, I have no design to touch upon it here. I shall only add, that the center of oscillation may be otherwise, very readily, computed, *from Corol. I.* even in cases where the forces acting on the bodies A, B, C have any given relation to each other. For, if a, b, c be taken to represent the, respective, measures of the said forces (or the *momenta* they would produce in a given time) it is evident, *from thence*, that the angular celerity that would be generated in the system (at the distance 1, from the center, during the same time) will be truly expressed by $\frac{a \times AP + b \times BP + c \times CP}{A \times \overline{AP}^2 + B \times \overline{BP}^2 + C \times \overline{CP}^2}$: which, in case of a

single body S, acted on by the force s , becomes $\frac{s \times SP}{S \times \overline{SP}^2}$ (or $\frac{s}{S \times \overline{SP}}$). Therefore, by putting this last value equal to the

former, we have $SP = \frac{s}{S} \times \frac{A \times \overline{AP}^2 + B \times \overline{BP}^2 + C \times \overline{CP}^2}{a \times AP + b \times BP + c \times CP}$; shewing at what distance from the point of suspension the single body S must be placed, to acquire, by means of the force s , the same angular celerity as the system itself acquires, from the action of all the other forces given.

LEMMA.

If a given angle AOB be divided into two parts AOC, BOC, the product (or solid) contained under the square of the sine (CD) of the one part AOC, and the sine (CE) of the other BOC, will be a maximum, when the tangent FC of the former part is double the tangent GC of the latter, or when the sine of the difference of the parts, is one-third of the sine of the whole given angle.

Fig. 26. For, if the sine (CD) of the former part be denoted by x , and that (CE) of the latter by y , it is well known that ($\dot{x}$) the celerity of x 's increase (supposing C to move from A to B) will be in proportion to the celerity ($-\dot{y}$) of y 's decrease, as the co-sine

co-sine of FCD to the co-sine of GCE; that is, as $\frac{x}{FC}$ to $\frac{y}{GC}$. But, when x^2y is a *maximum*, we have $2xy + x^2\dot{y} = 0$, and consequently $\dot{x} : -\dot{y} :: x : 2y$. Hence, by equality, $\frac{x}{FC} : \frac{y}{GC} :: x : 2y$; and therefore $FC = 2GC$. Let, now, OH be drawn to bisect FC in H, and let HM and GN be perpendicular to FO; then, since it is proved that $FC = 2GC$, it follows that $FH = \frac{1}{2}FG$, and that HM, by similar triangles, must likewise be $= \frac{1}{2}GN$: but HM and GN are sines of the angles HOM and GO, to the equal radii OH and OG; whence the latter part of the *Lemma* is also manifest.

PROBLEM II.

Suppose that a plane ABC, moving with a velocity and direction represented by bB, is acted on by a medium, or fluid, whose particles move with a velocity represented by DB, and in directions parallel thereto; to determine the effect of the fluid on the plane, in the direction of its motion BH, and also what the angle of inclination ABD must be, that the effect may be the greatest possible.

Because a particle, impinging on the plane at B, moves thro' the space DB in the time that the plane itself, from *abc*, arrives at the position ABC, it is evident that the distance (De) of the said particle from the plane (produced), at the beginning of that time, will be the measure of the relative celerity where-with the particles of the fluid approach the plane in a direction perpendicular thereto; and, consequently, that the force of the stream in that direction, will be as $\overline{De}$ (it being well known that the force of a stream upon any plane-surface, is always as the square of the relative celerity with which the particles approach it, in a perpendicular direction). Hence, by the resolution of forces, it will be, as the radius, is to the sine of the angle ABH (or *abH*), so is the force $\overline{De}$, to its required efficacy in the proposed direction BH.

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Moreover, with regard to the latter part of the Problem, the angle bBD , which the directions of the two motions make with each other, being given, as well as the sides Bb , BD containing it, the remaining angle BbD will from thence be known, as likewise Db : and so De being the sine of the angle Dbe , to the given radius Db , the effect $(\overline{De})^2 \times \sin. abH$ will therefore be a *maximum*, when $\sin. Dbe \times \sin. abH$ is a *maximum*; that is (*by the Lemma*), when the sine of the difference of the angles Dbe , abH , is equal to $\frac{1}{2}$ part of the sine of the whole given angle BbD : from whence the difference being given, the angles themselves will be known.—The geometrical construction from hence, is extremely easy; for, having from the center b , with any radius, described the arch mr , on rb produced (if necessary) let fall the perpendicular mp ; take $pq = \frac{1}{2}$ of mp , and draw qs parallel to pr , cutting the circle in s ; then bisect the arch ms by the line bac , and the thing is done: for the sine sv of Sr (or of the difference of the angles Dbe , abH) is by construction $(= pq) = \frac{1}{2}$ of mp the sine of the whole given angle BbD ; as it ought to be, *by the Lemma*.

But, if you had rather have a general Theorem expressed in algebraic terms, then let the velocity (bB) of the plane be put $= a$, and that (DB) of the fluid $= b$; and let the sine, and co-sine of the given angle DBb (to the radius 1) be denoted by m , and n , respectively; also, having drawn BFL perpendicular to bFe , put $bF = x$, and $BF = y$; then, since FB and FL are tangents of the angles FbB and FbL , to the common radius bF , it appears, *by the Lemma*, that FL is $(= 2BF) = 2y$; whence (supposing LR and DQ to be perpendicular to BbQ , we have (by similar triangles) as $Bb(a) : BF(y) :: BL(3y)$
: $BR = \frac{3y}{a}$, and therefore $bR(BL - Bb) = \frac{3yy - aa}{a}$. Al-

so, $Bb(a) : bF(x) :: BL(3y) : LR = \frac{3xy}{a}$. But the value of DQ being mb , and that of $bQ = nb - a$, we have again, by sim. triang.) $mb : nb - a :: \frac{3xy}{a} : \frac{3yy - aa}{a}$, and consequently

$$3yy - aa = \frac{nb - a}{mb} \times 3xy, \text{ or } 3yy - yy - xx = \frac{nb - a}{mb} \times 3xy \quad (\text{be-})$$

(because $yy + xx = aa$); whence $\left[\frac{x}{y}\right]^2 + \frac{nb-a}{nb} \times \frac{3x}{y} = 2$, and from thence, by completing the square and extracting the root, $\frac{x}{y} = \sqrt{2 + \frac{9}{4} \times \frac{nb-a}{nb}} - \frac{3}{2} \times \frac{nb-a}{nb}$, equal to the tangent of the angle bBF , the complement of the required angle abH , or ABH . $\mathcal{Q}. E. I.$

COROLLARY I.

If the given angle DBb be a right-one (which is the case when regard is had to the wind striking against the sails of a windmill); then, m being $= 1$, and $n = 0$, our expression for the tangent of bBF (which here is equal to the angle of inclination ABD) will become $\sqrt{2 + \frac{9aa}{4bb} + \frac{3a}{2b}}$; and this, if a be taken $= 0$, or the plane be supposed at rest, will be $= \sqrt{2}$, barely; answering to an angle of $54^\circ 44'$. But if the velocity of the plane be supposed $\frac{1}{2}$, $\frac{1}{4}$, or $\frac{1}{8}$ of the velocity of the medium or stream, then the angle of inclination ABD will be found from hence equal to $58^\circ 14'$, $61^\circ 27'$, or $66^\circ 58'$, respectively; so that, the greater the velocity of the plane is, the greater also will be the angle of inclination. Hence it appears that the sails of a windmill, that the effect may be the greatest, ought to be more turned towards the wind in the extrem parts where the motion is swiftest, than in the parts nearer to the axis of motion; in such sort, that the tangent of the angle formed by the direction of the wind and the sail, may be, every-where, equal to $\sqrt{2 + \frac{9aa}{4bb} + \frac{3a}{2b}}$; the velocity a being proportional to the distance from the axis of motion.

COROLLARY II.

If, instead of the angle DBH (or DB $\acute{b}$), the angle DBA, which the direction of the stream makes with the plane, be given; then it will appear, that the effect will, in this case, be a *maximum*, when the sine of the angle ABH, made by the plane and the direction of its motion, is to the sine of the said given angle DBA, in the given proportion of $\frac{1}{3}$ BD to B $\acute{b}$. For, the force in the perpendicular direction FB being expressed by $\overline{Dd}^1$, its effect in the direction BH will, therefore, be defined by $\overline{De}^2 \times \frac{BF}{B\acute{b}}$, or its equal $\frac{\overline{Dd}^1 \times Ee}{B\acute{b}}$ (supposing BA produced to meet D $\acute{c}$ E in E). Now DB and the angle DBE (as well as B $\acute{b}$) being supposed given, DE is given from thence. But it is well known, that the square of one part of a given line, drawn into the other part, will be a *maximum*, when the former part is the double of the latter. Consequently De must here be the double of Ee; which last, or its equal BF, will therefore be $= \frac{1}{3}$ DE.

But, sin. B $\acute{b}$ F : radius :: BF ($\frac{1}{3}$ DE) : B $\acute{b}$;

Add, radius : sin. DBA :: BD : DE;

by compounding of which, we have the proportion above laid down. But that proportion, it may be observed, can only take place when B $\acute{b}$ is equal to, or greater than $\frac{1}{3}$ of DE: for, when B $\acute{b}$ is less than $\frac{1}{3}$ of DE, Ee (which is always less than B $\acute{b}$) cannot be equal to $\frac{1}{3}$ of DE; but will approach the nearest to it, when BF coincides with B $\acute{b}$, that is, when the angle F $\acute{b}$ H, or ABH is a right-one; and in this case, the effect will be a *maximum*, when the direction of the motion is perpendicular to the plane.—If the given angle DBA be a right-one (which position appears from hence to be the most advantageous, because DE then becomes = DB) it follows that the sine of the angle ABH, which the required direction makes with the plane, will be to the radius, as $\frac{1}{3}$ part of the velocity of the stream is to the velocity of the plane (or sail).—Hence, if the force of the wind be capable of producing a degree of celerity in a ship, greater than $\frac{1}{3}$ part of its own celerity, it is evident that the ship may run swifter up-
on

on an oblique course, than when she sails directly before the wind*.

PROBLEM III.

Suppose that a thread $ACnCA$, having two equal weights A , A , suspended at the ends thereof, is hung over two tacks C , C , in the same horizontal line; and that to the middle point of the thread (n) equally distant from the tacks, another given weight B is fixed, which is permitted to descend by its own gravity, so as to cause the other two weights, at the same time, to ascend: it is proposed to find the law of the velocity by which the said weights ascend and descend; abstracting from the resistance of the air, the weight of the thread, and the friction on the tacks.

Let v denote the velocity of B (measured by the distance Fig. 28. that might be uniformly gone over in one second of time), and let b ($= 32\frac{1}{8}$ feet) $=$ the measure of the velocity which gravity can generate in a falling body, in one second; putting $CE = a$, $En = x$, $Cn = y$, and the tension of the thread $= w$: then $\frac{\dot{x}}{v}$ being the time in which B would, uniformly, describe the distance $\dot{x}$, we shall have, as 1 (second) is to $\frac{\dot{x}}{v}$, so is b (the velocity generated by gravity in one second) to, $\frac{b\dot{x}}{v}$, the velocity generated (or destroyed) by gravity in the time $\frac{\dot{x}}{v}$.

Moreover it will be, as BC (y) : En ($\sqrt{yy - aa}$) :: v : $\frac{v\sqrt{yy - aa}}{y}$ $=$ the velocity with which the weights A , A ascend; whose fluxion $\frac{\overline{yy - aa} \times \dot{y} + aav\dot{y}}{yy\sqrt{yy - aa}} (= \frac{\overline{yy - aa} \times \dot{y} + aav\dot{y}}{yyx})$ is therefore the increase of that velocity, in the time $\frac{\dot{x}}{v}$: but were

* In the above considerations the velocity of the plane (or sail) is, all along, treated as a given quantity; because the same direction that gives the effective force the greatest, when the velocity is given, must necessarily give the velocity the greatest possible, when the force, alone, is given.

not the string to act on the said weights, their velocity (instead of being increased) would be diminished, and that by the quantity $\frac{b\dot{x}}{v}$ (as is found above). Therefore the whole alteration of motion arising from the tension of the string is to that arising from the action of gravity, in the proportion of

$\frac{b\dot{x}}{v} + \frac{yy - aa \times y\dot{v} + aav\dot{y}}{yyx}$ to $\frac{b\dot{x}}{v}$; and, consequently, the tension of the string (w) will be to the weight of the body A, in the same proportion: whence we have $w = A \times 1 + \frac{yy - aa \times y\dot{v} + aav\dot{y}}{byyx}$.

Again, it will be (by the resolution of forces) as $Cn(y)$ is to $En(x)$, so is $2w$ (the double of the tension of the thread) to $\frac{2wx}{y}$, the effect of that tension to retard the descent of the weight B; which being subtracted from the gravity (B), the remainder $B - \frac{2wx}{y}$ will be the force by which B's motion is accelerated. Hence we have, as B is to $B - \frac{2wx}{y}$, so is $\left(\frac{b\dot{x}}{v}\right)$ the velocity that would be generated by the gravity in the time $\frac{\dot{x}}{v}$, to that ($\dot{v}$) generated in the same time, by the force $B - \frac{2wx}{y}$. From whence, by multiplying extremes and means,

we get $v\dot{v} = b\dot{x} - \frac{2bxxw}{By} = b\dot{x} - \frac{2bAx\dot{x}}{By} - 2A \times \frac{yy - aa \times y\dot{v} + a^2v^2\dot{y}}{By^3}$

(by substituting the value of w) $= b\dot{x} - \frac{2bA\dot{y}}{B} - \frac{2A\dot{v}\dot{v}}{B} + \frac{2a^2A}{B} \times \frac{y\dot{v} - v^2\dot{y}}{y^3}$ (because $yy = x\dot{x}$) $= b\dot{x} - \frac{2bA\dot{y}}{B} - \frac{2A\dot{v}\dot{v}}{B} + \frac{a^2A}{B} \times \frac{2y^2v\dot{v} - 2v^2y\dot{y}}{y^4}$: and, consequently, by taking the fluent,

$\frac{v^2}{2} = b\dot{x} - \frac{2bAy}{B} - \frac{Av^2}{B} + \frac{a^2A}{B} \times \frac{vv}{yy} + d$ (d being the necessary correction); which equation, if $\frac{B}{2A}$ be put $= m$, will be reduced to $v = y\sqrt{\frac{2bm\dot{x} - 2by + 2dm}{m + 1.yy - aa}}$; shewing the true velocity

locity of the body B; whence that of the body A ($= x\sqrt{\frac{2bm x - 2by + 2dm}{m + 1 \cdot yy - aa}}$) will also be known. Q. E. I.

COROLLARY I.

If the first values of x and y , when the motion commences, be expressed by f and g , respectively; then, v being $= 0$, when $x = f$, and $y = g$, we shall have $0 = 2bmf - 2bg + 2dm$, and consequently $2dm = -2bmf + 2bg$; so that the general va-

lue of v will be $y\sqrt{\frac{2bm x - 2by - 2bmf + 2bg}{m + 1 \cdot yy - aa}} = y\sqrt{\frac{2bm x - 2by - 2bmf + 2bg}{m + 1 \cdot yy - aa}}$.

From whence the greatest distance through which the weight B can descend, before its whole motion is destroyed by the other weights A, A, may be easily determined: for, since the velocity, at the lowest point of the descent, vanishes, or becomes equal to nothing, we shall, in that circumstance, have

$2bm \times x - f - 2b \times y - g = 0$, or $m \times x - f + g (= y = \sqrt{xx + aa}) = \sqrt{xx + gg - ff}$; which, squared, gives

$m^2 \times x - f + 2mg \times x - f = xx - ff$: whence, dividing by $x - f$, we have $m^2 \times x - f + 2mg = x + f$; and consequently $x = \frac{2mg - 1 + mm \cdot f}{1 - mm}$; exhibiting the distance of the

point n below the horizontal line CC, when the whole motion is destroyed, and all the weights begin to move the contrary way. But it must be observed, that this can only happen when m is less than unity, or when the weight B is less than the sum of the other two: for, if m be equal to unity, x will be infinite; and, if m be greater than unity, the value of x will come out negative; which shews the thing to be impossible, or that the weight B must continually descend; except when m is less than unity, or B less than $2A$: in which last case, it appears that the bodies will oscillate, backwards and forwards, continually; in such sort, that the two extrem distances from the horizontal line CC will be expressed by f and $\frac{2mg - 1 + mm \cdot f}{1 - mm}$;

whereof the latter, when $f = 0$, will become $\frac{2ma}{1 - mm} =$

$\frac{m}{1 - mm} \times CC$; shewing the lowest descent of n , from the line CC , when the motion commences from that line.—By making f and $\frac{2mg - 1 + mm \cdot f}{1 - mm}$ equal to each other, we get $f = mg = \frac{B}{2A} \times g$, or $f : g :: B : 2A$. From which it appears, that, if the first position of the weight B be such, that En is to Cn in the given proportion of B to $2A$, no motion at all will ensue, but the weights remain in *equilibrio*. Whence it is evident, that, if the motion commences from any point below that here determined, the weight B will first of all ascend, till the distance from CC is $\frac{2mg - 1 + mm \cdot f}{1 - mm}$; after which it will again descend, to its first distance f ; and so on, backwards and forwards, continually.

COROLLARY II.

If CnC , in the first position of B , be supposed to coincide with the horizontal-line CEC , and the body B be impelled from thence with any given celerity c (measured, as above, by the space that would be uniformly gone over in one second of time); then, v being $= c$, when $x = 0$ and $y = a$, we shall, by substituting these values in the general equation

$$(v = y \sqrt{\frac{2bmx - 2by + 2dm}{m + 1 \cdot yy - aa}}) \text{ obtain } c = a \sqrt{\frac{-2ba + 2dm}{maa}} =$$

$$\sqrt{\frac{2dm - 2ba}{m}}, \text{ and consequently } 2dm = mc^2 + 2ba; \text{ so that } v$$

$$\text{is here } = y \sqrt{\frac{2bmx - 2by + 2ba + mc^2}{m + 1 \cdot yy - aa}}; \text{ which, when } b = 0, \text{ or,}$$

when the bodies are not acted on by gravity, will become $v =$

$$\frac{m^{\frac{1}{2}}cy}{\sqrt{m + 1 \cdot yy - aa}}. \text{ And in this case, the time of moving thro' } En$$

$$(\text{whereof the fluxion is } \frac{\dot{x}}{v} = \frac{\dot{x} \sqrt{m + 1 \cdot yy - aa}}{m^{\frac{1}{2}}cy} = \frac{\dot{x} \sqrt{m + 1 \cdot xx + ma^2}}{m^{\frac{1}{2}}c \sqrt{xx + aa}})$$

may be readily found, by means of an *hyperbola*, whose transverse

verse and conjugate axes are $\frac{2a}{\sqrt{m}}$ and $2a$; it being in proportion to the time of moving uniformly over the same distance (x) with the given celerity at E, as the arch of the *hyperbola* is to its ordinate x .

PROBLEM IV.

Supposing a spherical body of ice, or any other matter, revolving about its axis, to be reduced to a state of fluidity; to determine the change of figure thence arising.

It is demonstrable, that the figure of an homogeneous fluid, Fig. 29. revolving about an axis (PS), having all its particles quiescent with regard to each other, must be *that* of an oblate spheroid OAPES (*see Art. 395 of my Doctrine of Fluxions*); and that the particular species of such spheroid, answering to any given time of revolution p , will be truly defined by the equation

$$p = q \sqrt{\frac{t^3}{3 + u \times A - 3t}}; \text{ where } 1 : 1 + tt :: PS^2 : AE^2; \text{ PS be-}$$

ing the axis, and AE the equatoreal diameter; also A = the circular arch, whose radius is unity, and tangent t ; and q = the time wherein a solid sphere, of the same magnitude and density with the spheroid, must revolve, so that the centrifugal force at the equator thereof, may be exactly equal to the attraction, or gravity. Now it is evident, that, whatsoever figure a fluid, revolving about an axis, at any time hath, the momentum of rotation about the axis will be no-ways changed, with the figure, by the action of the particles on each other: so that the momentum of our proposed fluid, arising from the sphere of ice, will, at all times, be the very same with that of the sphere itself. From whence it may be easily proved, that the time wherein one intire revolution of the fluid, considered as a spheroid, might be uniformly performed, must be always as AE^2 : therefore, if we make $e = AE$, and put $d =$ the diameter of the sphere (or of the fluid, when $AE = PS$) it follows that the said time * will be truly expressed

* At Art. 399. of my Fluxions, this time is, by mistake, put down $= \frac{e}{d} \times s$ (instead of $\frac{e^2}{d^2} \times s$); whereby the remaining part of that Article is rendered erroneous. by

by $\frac{ee}{dd} \times s$ (supposing s to denote the given time of revolution of the body, when under the form of a sphere): which being put $(= q\sqrt{\frac{3t^3}{3+u \times A - 3t}})$ the time wherein the revolution is performed, when the particles are in *equilibrio*, we shall thence have $\frac{3+u \times A - 3t}{t^3} = \frac{2g^2}{3t^2} \times \frac{d^4}{e^4}$.

But, because PS $(= \frac{AE}{\sqrt{1+u}}) = \frac{e}{\sqrt{1+u}}$, the mass of the spheroid will therefore be as $\frac{e^3}{\sqrt{1+u}}$ $(= AE^2 \times PS)$; and *that* of the sphere, as d^3 : which two quantities being made equal to each other, we have $\frac{d^4}{e^4} = \frac{1}{1+u}^{\frac{3}{2}}$. And, this value being

wrote in the room of its equal, we have $\frac{3+u \times A - 3t}{t^3} = \frac{2g^2}{3t^2} \times \frac{1}{1+u}^{\frac{3}{2}}$, or $\frac{3+u \times A - 3t}{t^3} \times 1+u^{\frac{3}{2}} = \frac{2g^2}{3t^2}$. From the

resolution of which equation the value of t , and the spheroid itself, will be known. The spheroid thus determined, is *that* under which the fluid might remain in *equilibrio*, were the particles to be, once, quiescent with respect to each other: but the particles, in their recess from the axis, do, through the centrifugal force, acquire a motion from the axis, which is not immediately destroyed, on the fluid's assuming the figure, or degree of oblateness above determined; the equatoreal parts still continuing to recede from the axis, till the gravitation, by degrees, prevails, and in the end quite overcomes the said motion. After which the equatoreal parts will begin to subside, and again approach the axis, in the very same manner they before receded therefrom: and so will continue oscillating, backwards and forwards, *ad infinitum*. But if the fluid is supposed to have some degree of tenacity, the oscillations will be, every time, contracted, and the parts of the fluid will then converge to an equilibrium, under the form above determined.

L E M-

LEMMA.

Supposing a body to move with an uniform celerity, in a right-line AD; to determine the rate of increase of the relative celerity by which it recedes from a given point C, out of that line.

Make CA (perpendicular to AD) = a , and AB = x ; and let the measure of the body's celerity, or the space gone over in a given time g , be denoted by c : then will $\frac{g\dot{x}}{c}$ express the time of describing $\dot{x}$ (or Bb); and it is well known, that $c \times \frac{x}{\sqrt{xx+aa}}$ ($= c \times \frac{AB}{BC}$) will be the true measure of the celerity with which CB increases; whose fluxion, $\frac{ca^2\dot{x}}{xx+aa}^{\frac{1}{2}}$, is therefore the (uniform) increase of that celerity, in the time $\frac{g\dot{x}}{c}$: hence it will be, as $\frac{g\dot{x}}{c}$: g (the time given) :: $\frac{ca^2\dot{x}}{xx+aa}^{\frac{1}{2}}$: $\frac{c^2a^2}{xx+aa}^{\frac{1}{2}}$ ($= \frac{c^2 \times AC^2}{CB^2}$) the required increase, that would uniformly arise in the given time g : which increase, since $\frac{c \times AC}{CB}$ represents the paracentric velocity of the body (in a direction perpendicular to CB) will be, always, expressed by the square of the measure of the body's paracentric velocity, applied to the distance (BC) from the given point, or center. *Q. E. I.*

COROLLARY.

It is evident from hence, that if a force, which in the given time g is sufficient to generate the said increase of velocity, be supposed to urge the body towards the center C, and thereby deflect it from its rectilinear motion, the celerity with which CB increases will *then* be uniform; because the force applied, each moment of time, is just sufficient to destroy the increase that would arise, in the same moment, from the body's being suffered to continue its motion uniformly in a right-line.—If the direction (Bb) of the motion is perpendicular to CB, the body, thus acted on (as no celerity is generated in the direction CB), will move in the circumference of a circle. Consequently

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the

the force above determined is the same with the centrifugal force in a circle, when the distance from the center, and the angular celerity, are the same.

But all this may be made to appear in a different manner, by supposing Bb exceeding small: for, if bE be made perpendicular to CB (produced), BE will then express the length whereby CB would be uniformly augmented, in the time $\left(\frac{g \times Bb}{c}\right)$

of describing Bb ; and therefore eb , the excess of Cb above CE , will be the space through which the force must cause the body to descend, in order that the increase of the distance from the center C may be the same as would uniformly arise with the first celerity, at B . But it is evident that this excess eb

(which, by the property of the circle, is $= \frac{Eb^2}{2CE}$, or $\frac{Eb^2}{2CB}$)

also expresses the effect of the force, necessary to cause a body to revolve in the circumference of a circle Ecf , with the same angular celerity.—To determine, from hence, the velocity which this force would generate in the given time g , we have,

as $\frac{g^2 \times \overline{Bb}^2}{cc}$ (the square of the time of describing Bb , or Be) is to

g^2 , so is $\frac{\overline{Eb}^2}{2CB}$ to $\left(\frac{c^2 \times \overline{Eb}^2}{2CB \times \overline{Bb}^2}\right)$ the space through which the

ball might fall, by means of the said force, in the given time g ; the double of which, $\frac{c^2 \times \overline{Eb}^2}{CB \times \overline{Bb}^2}$ (or its equal $\frac{c^2 \times \overline{AC}^2}{CB \times \overline{CB}^2}$)

is, therefore, the true measure of the velocity sought; because the distance gone over by a falling body is but the half of that which might be described in the same time, with the velocity acquired at the end of the descent.—The quantity here determined (as has been before observed) is the measure of the force by which the body is made to recede from C with an uniform celerity: if a force, less or greater than this, be supposed to act, the difference will cause an increase or decrease of celerity in the line CB , proportional to the said difference.

PROBLEM V.

Suppose that a body, let go from a given place A, in a given direction, with a given celerity, is continually solicited towards a given point C, by a given centripetal force; to determine the path ABP in which the body will move.

From the center C, through A, let the circumference of a Fig. 31. circle ADK be described; and, supposing B to represent the place of the body,

put $\left\{ \begin{array}{l} \text{the radius CD of the circle} \dots\dots\dots = a, \\ \text{the radius vector CB} \dots\dots\dots = x, \\ \text{the arch AD, measuring the angle ACB} \dots\dots = z, \\ \text{the time of describing the angle ACB} \dots\dots = t, \\ \text{the meas. of the celerity with which the line CB incr.} = v, \\ \text{the meas. of the celer. with which the area ACB incr.} = au, \\ \text{the measure of the centripetal force} \dots\dots\dots = Q; \end{array} \right.$

where, by the measure of a celerity, I mean the space that would be uniformly described with that celerity in a given time g ; and by the measure of a force, I understand the measure of the celerity that might be uniformly generated by the force, in the same given time.

Since the celerity with which the area ACB increases is expressed by au , it is evident that the paracentric velocity of the radius vector CB, at the middle point (b), will be expressed by $\frac{au}{x}$, and that of the body itself by $\frac{2au}{x}$; the square of which last, divided by x , will give $\frac{4a^2u^2}{x^3}$ for the true measure of the centrifugal force (*by the Lemma*); whence it appears that $\left(\frac{4a^2u^2}{x^3} - Q\right)$ the excess thereof above the centripetal force Q , is that force whereby the celerity v is accelerated: therefore we have $g : t :: \frac{4a^2u^2}{x^3} - Q$ (the measure of the celerity generated in the given time g) : $\dot{v}$. But, because the paracentric velocity of the body is $\frac{2au}{x}$, that of the point D (describing the

circular arch AD) will be $\frac{2au}{x} \times \frac{a}{x}$, or $\frac{2a^2u}{xx}$; and so we have $g : \dot{t} :: \frac{2a^2u}{xx}$ (the distance described in the time g) : $\dot{z}$ (the distance described in the time $\dot{t}$): whence, by equality, $\frac{4a^2u^2}{x^3} - Q$
 $:\dot{v} :: \frac{2a^2u}{xx} : \dot{z}$; and consequently $\dot{v} = \frac{4a^2u^2}{x^3} - Q \times \frac{x^2\dot{z}}{2a^2u}$.

Again, the spaces $\dot{z}$ and $\dot{x}$ (described in the same time) being in the same proportion as the celerities $\frac{2a^2u}{xx}$ and v , with which they are described, we also have $v = \frac{2aau\dot{x}}{xx\dot{z}}$. To exterminate v and $\dot{v}$ out of *this*, and the preceding equation, make $\frac{a\dot{x}}{xx} = \dot{w}$, (or $w = -\frac{a}{x} + 1$); then $v = \frac{2aau\dot{w}}{\dot{z}}$, and $\frac{2aau\dot{w} + 2aau\dot{w}}{\dot{z}} = \dot{v} = \frac{4a^2u^2}{x^3} - Q \times \frac{x^2\dot{z}}{2a^2u} = \frac{2u\dot{z}}{x} - \frac{Qx^2\dot{z}}{2a^2u} = \frac{2u\dot{z} \times 1 - w}{a} - \frac{Q\dot{z}}{2u \times 1 - w}$; (by writing $\frac{a}{1-w}$ for its equal x); which equation may be reduced to $\frac{\ddot{w}}{\dot{z}\dot{z}} + \frac{w}{a} = \frac{1}{aa} - \frac{\dot{w}}{u\dot{z}} - \frac{Q}{4au^2 \times 1 - w}$; expressing the general relation of w and z , or of x and z , according to any value of u . But in the case propounded, wherein no force is supposed to act, besides that tending to the center C, the celerity au with which the area ACB increases, will be a constant quantity; and therefore, $\dot{u}$ being here $= 0$, our equation becomes $\frac{a^2\ddot{w}}{\dot{z}\dot{z}} = 1 - w - \frac{aQ}{4u^2 \times 1 - w}$; from whence, when Q is given in terms of x (or w), the relation of w and z may be determined.

COROLLARY I.

Hence, if the centripetal force, by which a body describes a given orbit at rest, be known, the increase of that force, when the orbit itself is supposed to have a motion round the center of force, may be easily deduced: for, let the angular motion of the orbit, be to that of the body in the orbit, in the constant ratio of m to 1 ; then, the whole angular celerity of the

the body, here, being in proportion to the angular celerity when the orbit is quiescent, as $m + 1$ to 1, the centrifugal force here, will therefore be to that $\left(\frac{4a^2u^2}{x^3}\right)$ in the quiescent orbit, in the duplicate ratio of $m + 1$ to 1 (*by the Lemma*), and so will be truly expressed by $\overline{m + 1}^2 \times \frac{4a^2u^2}{x^3}$. From whence it appears, that $\overline{mm + 2m} \times \frac{4a^2u^2}{x^3}$ is the increase of the centrifugal force arising from the motion of the orbit: which quantity, therefore, must be *that* whereby the centripetal force ought to be likewise increased, in the moveable orbit; so that the difference of the two forces, whereby the motion of the body in the line CB is accelerated, may be the same here, as in the quiescent orbit; in which case the value of CB itself, in all contemporay positions, must necessarily be the same. Hence it appears, that the increase of the centripetal force, in order to the description of a moveable orbit, will be always inversely as the cube of the distance; and will, moreover, be to the centrifugal force in the quiescent orbit (in all contemporary positions), in the constant ratio of $mm + 2m$ to 1.

COROLLARY II.

If the centripetal force (Q) be supposed, inversely, as the square of the distance, and the given value thereof, at the lower apse A, be to the centrifugal force there, in any given ratio of $1 - e$ to 1; then, as the general value $\left(\frac{4a^2u^2}{x^3}\right)$ of the centrifugal force, will, at A, become $= \frac{4u^2}{a}$, the centripetal force there will be expressed by $\frac{4u^2}{a} \times \overline{1 - e}$; and consequently that at B by $\frac{\overline{1 - e} \times 4u^2}{a} \times \frac{a^2}{x^3}$, or its equal $\frac{\overline{1 - e} \times 4u^2}{a} \times \overline{1 - w}^2$. Which value being substituted for Q, our equation here becomes $\frac{a^2\ddot{w}}{xz} = e - w$: whence, multiplying by $\dot{w}$ and taking the fluent, we get $\frac{a^2\dot{w}^2}{2xz} = ew - \frac{w^2}{2}$ (where, the angle CAB being supposed

Fig. 32. supposing a right-one, no correction is necessary); so that we

have $\dot{z} = \frac{a\dot{w}}{\sqrt{2ew - ww}}$, or $\frac{e\dot{z}}{a} = \frac{e\dot{w}}{\sqrt{2ew - ww}}$: but the last of these quantities is known to express the fluxion of a circular arch (A), whose versed-sine is w and radius e ; therefore $\frac{e\dot{z}}{a}$ being $= \dot{A}$, or $a : e :: z : A$, it follows that the arches z and A (which are in the same proportion with their radii a, e) must be similar, and consequently their versed-sines, AF and w , in the same proportion above specified, or as a to e : whence we have $w = \frac{e}{a} \times AF$, that is, $1 - \frac{AC}{BC} = e \times \frac{AF}{AC}$. But $\frac{AF}{AC} = \frac{AC - CF}{AC} = 1 - \frac{CF}{CD} = 1 - \frac{CE}{CB}$ (supposing BE perpendicular to AC);

therefore $1 - \frac{AC}{BC} = e \times 1 - \frac{CE}{CB}$, and consequently $BC - AC (= BD) = e \times CB - CE$; from which equal quantities take away $e \times BD$, so shall $BD - e \times BD = e \times AE$, and therefore $1 - e : e :: AE : BD$; which is a known property of the conic sections, with respect to lines drawn from the focii. Hence it appears, that the trajectory will be an *ellipse*, *parabola*, or *hyperbola*, according as the antecedent $1 - e$ is greater, equal to, or less than the consequent e ; or according, as the centripetal force at A , is greater, equal to, or less than half the force sufficient to retain the body in the circular orbit ADK .—As to the particular species of the curve, corresponding to any given value of e , it is, from hence, very easily determined: for, if AO be made to represent the semi-transverse axis, then will $AO : OC (:: AE : BD, p. conics) :: 1 - e : e$; therefore, by division, $AO : AC :: 1 - e : 1 - 2e$; whence AO is known.

COROLLARY III.

If to the foregoing force, varying in the inverse ratio of the square of the distance, another force, which is inversely as the cube of the distance, be joined (which, at A , is to the former part in any given ratio of s to $1 - e$), the place of a body, thus acted on, may be found in the same conic section

Fig. 33. $APRS$, above determined, supposing it to have a motion about

about the focus C, which is to that of the body in the section (referred to the same point C) in the constant ratio of m to 1; the value of m being $= \sqrt{1+s} - 1$, or such, that $mm + 2m : 1 :: s : 1$ (by *Corol. I.*)—If the centripetal force be barely as the cube of the distance inversely, the curve A'B will degenerate to a right-line; in which the body will continue to move with an uniform velocity, while the line itself BA' (always touching the circle in A') is so carried along by the motion of the radius CA', that the angle ACA' shall be to the angle A'CB, in the constant proportion above specified; the ratio of the centripetal, and centrifugal forces at A (and consequently in every other position) being expressed by that of s to $1+s$.

COROLLARY IV.

If the centripetal force to be as any power of the distance, whose exponent is n , and the given value thereof, at A, be in proportion to the centrifugal force $\left(\frac{4u^2}{a}\right)$, as r to 1;

we shall then have $Q = \frac{4ru^2}{a} \times \frac{x^n}{a^n} = \frac{4ru^2}{a} \times \frac{1}{1-w}^n$; and here

our equation, $\frac{a^2 \ddot{w}}{z \dot{z}} = 1 - w - \frac{aQ}{4u^2 \times 1-w}^2$, will become

$$\frac{a^2 \ddot{w}}{z \dot{z}} = 1 - w - \frac{r}{1-w}^{n+1}; \text{ which being multiplied by } \dot{w},$$

and the fluent taken, we thence get $\frac{a^2 \dot{w}^2}{2z \dot{z}} = w - \frac{w^2}{2} -$

$$\frac{r \cdot 1-w}^{n+1} \frac{1}{n+1} + \frac{r}{n+1}, \text{ and consequently}$$

$$\dot{z} = \frac{a \dot{w}}{\sqrt{2w - w^2 - \frac{2r \cdot 1-w}{n+1}^{n+1} + \frac{2r}{n+1}}}; \text{ from whence the}$$

value of z , by infinite series, or the quadrature of curves, may be found.

But when r differs but little from unity, and the orbit is nearly circular, $\frac{r}{1-w}^{n+1}$ (because of the smallness of w)

will

will be nearly equal to $r \times \frac{1+n+2}{n+3} \cdot w$; and therefore $\frac{a^2 \ddot{w}}{z \ddot{z}} = 1 - w - r - \frac{1+n+2}{n+3} \cdot r w = 1 - r - \frac{1-r}{n+3} \cdot w$, nearly; and consequently $\frac{a^2 \ddot{w}}{n+3 \cdot z \ddot{z}} = \frac{1-r}{n+3} - w$.

Fig. 33. Put $f = \frac{1-r}{n+3}$, and $A = \sqrt{n+3} \times z$, that is, let A represent an arch A'D of the circle ADK, which is always to the arch AD (or z), in the constant ratio of $\sqrt{n+3}$ to 1: then AA' being $= n+3 \times z \ddot{z}$, our equation, by substituting these values, will become $\frac{a^2 \ddot{w}}{AA'} = f - w$; which differs in nothing from that $\left(\frac{a^2 \ddot{w}}{z \ddot{z}} = e - w \right)$ resolved in *Corol. II*, excepting only, that A and f are here used, instead of z and e : whence it is manifest, that the value of w (there represented by $e \times$ versed-sine of z) will here be truly expressed by $f \times$ versed-sine of A : from whence and what is there demonstrated, it also appears, that the place of the body will be in the periphery of a given ellipse A'BR, revolving about its focus C, with an angular celerity, which is to that of the body in the ellipse, in the constant ratio of the arch AA' to the arch A'D, or as $1 - \sqrt{n+3}$ to $\sqrt{n+3}$. And it is evident, that the motion of the apfides will be to the motion of the body in the ellipse, referred to the focus C, in the same given ratio; and that the angle described by the body in moving from one apside to the other (because AD is always $= A'D \times \frac{1}{\sqrt{n+3}}$) will be $= 180^\circ \times \frac{1}{\sqrt{n+3}} = \frac{180^\circ}{\sqrt{n+3}}$. All which conclusions, as well as those derived in the preceding *Corollaries*, exactly agree with what Sir ISAAC NEWTON has demonstrated, by a very different method, in the third and ninth *Sections* of the first book of his *Principia*.—As to the motion of the *apfides* of the lunar orbit, with the other inequalities depending on the sun's action, these require the use of other principles, and the solution of the following

PROBLEM VI.

The same being supposed as in the last Problem, and that, besides the force tending to the center C, another force, whose measure is R, acts continually on the body, in a direction perpendicular to the radius-vector BC; it is proposed to determine the curve ABP which the body, so acted on, will describe.

Every thing in the preceding Problem being retained, Fig. 31. we have nothing more to do here, than to get an equation for u , by means of the new force, whereon the increase or decrease of u intirely depends. In order to this, we

have, as g (the given time) is to $\dot{t}$, so is R, the velocity generated in the time g , to $\frac{R\dot{t}}{g}$, the velocity generated in the time

$\dot{t}$, in a direction perpendicular to BC: whence the corresponding increase of the celerity au , with which the area

ACB is generated, will be expressed by $\frac{R\dot{t}}{g} \times \frac{x}{2}$, that is, $\frac{R\dot{t}}{g} \times \frac{x}{2}$

will be $= a\dot{u}$. But, as it has been proved that $g : \dot{t} :: \frac{2a^2u}{xx} : \dot{z}$,

we have $\frac{\dot{t}}{g} = \frac{x^2\dot{z}}{2aau}$; and therefore $a\dot{u} = \frac{Rx^2\dot{z}}{4aau}$, or $2u\dot{u} =$

$\frac{\frac{1}{2}Rx^2\dot{z}}{a^3} = \frac{\frac{1}{2}R\dot{z}}{1-w}$ (because $x = \frac{a}{1-w}$). Hence, by taking the

fluent, we have $u^2 = c^2 + \text{flu.} \frac{\frac{1}{2}R\dot{z}}{1-w}$ (c^2 being put for the necessary correction, or the value of u^2 when $z=0$). From

which equation, and that $\left(\frac{\ddot{w}}{zz} + \frac{w}{aa} = \frac{1}{aa} - \frac{\dot{u}\dot{w}}{u^2z} - \frac{Q}{4au^2 \times 1-w} \right)$

derived by the preceding Problem, the relation of u , w , and z may be determined, when the law of the forces Q and R is assigned. Q. E. I.

COROLLARY

If the forces $\mathcal{Q}$ and R are supposed to be in proportion to $\left(\frac{4cc}{a}\right)$ the centrifugal force at A , as Δ to 1 , and Π to 1 , respectively (Δ and Π being any variable quantities whatever), and if the celerity (au) with which the area ACB increases be supposed in proportion to (ac) the first value thereof at A , as $\Sigma^{\frac{1}{2}}$ to 1 ; then, Q being $= \frac{4cc\Delta}{a}$, $R = \frac{4cc\Pi}{a}$, and $u = c \Sigma^{\frac{1}{2}}$,

our two equations, by substituting these values, will become $\Sigma = 1 + \text{flu.} \frac{2\Pi\dot{z}}{a \times 1 - w}$, and $\frac{\ddot{w}}{zz} + \frac{w}{aa} = \frac{1}{aa} - \frac{\dot{\Sigma}}{2\Sigma} \times \frac{\dot{w}}{zz} - \frac{\Delta}{a^2 \Sigma \times 1 - w}$;

or $\Sigma = 1 + \text{flu.} \frac{2\Pi\dot{z}}{1 - w}$, and $\frac{\ddot{w}}{zz} + w = 1 - \frac{\dot{\Sigma}}{2\Sigma} \times \frac{\dot{w}}{zz} - \frac{\Delta}{\Sigma \times 1 - w}$, by

making AC unity; which last equation will be rendered still more commodious, by writing for $\dot{\Sigma}$ its equal $\frac{2\Pi\dot{z}}{1 - w}$; whence

will be had $\frac{\ddot{w}}{zz} + w = \left(1 - \frac{\Pi\dot{w}}{\Sigma z \times 1 - w} - \frac{\Delta}{\Sigma \times 1 - w}\right) = \frac{1}{\Sigma}$

into $\Sigma - \Delta \times \frac{1}{1 - w} - \Pi \times \frac{\dot{w}}{z} \times \frac{1}{1 - w} = 0$: from which the

values of w and Σ may be found, when those of Δ and Π are assigned: by means whereof the time (t) of describing the angle z , will also be known, from the equation $\frac{\dot{t}}{g} = \frac{x^2 \dot{z}}{2aa u}$ (a-

bove derived): which by substituting $c \Sigma^{\frac{1}{2}}$ and $\frac{1}{1 - w}$ for their

equals u and $\frac{xx}{aa}$, gives $\dot{t} = \frac{g}{2c} \times \frac{z}{\Sigma^{\frac{1}{2}} \times 1 - w}$.

To exemplify the use of the equations here derived, by the resolution of a case on which the determination of the lunar orbit depends, let the force Δ , whereby the body is solicited towards the center, be considered, as composed of two parts; whereof the principal ($b \times \frac{1}{1 - w}$) is in the inverse duplicate-ratio of the distance; the other part, which is supposed small in comparison of the former, being as the distance $\left(\frac{1}{1 - w}\right)$ directly, drawn in-

to

to a series ($P' \times \text{cof. } pz + Q' \times \text{cof. } qz + R' \times \text{cof. } rz$, &c.) of co-
fines of multiples of the arch z , joined to small, given, coeffi-
cients P' , Q' , R' , &c. and let the force Π , acting in the per-
pendicular direction, be also supposed, as the distance directly,
drawn into a series ($P \times \text{fin. } pz + Q \times \text{fin. } qz + R \times \text{fin. } rz$ * &c.)
of fines of multiples of the same arch, joined to small, given,
coefficients P , Q , R , &c. According to these assumptions, by

substituting $b \times \overline{1-w}^3 - \frac{1}{1-w} \times \overline{P' \text{cof. } pz + Q' \text{cof. } qz}$ &c. and

$\frac{1}{1-w} \times \overline{P \text{fin. } pz + Q \text{fin. } qz}$ &c. for their equals Δ and Π , our

two equations, $\Sigma = 1 + 2 \text{ fluent } \frac{\Pi z}{1-w}^1$, and $\frac{\ddot{w}}{zz} + w =$

$\frac{1}{\Sigma} \times \Sigma - \Delta \times \overline{1-w}^{-2} - \Pi \times \frac{\dot{w}}{z} \times \overline{1-w}^{-3}$, will here become

$\Sigma = 1 + 2 \text{ fluent } P \dot{z} \text{fin. } pz + Q \dot{z} \text{fin. } qz$ &c. $\times \overline{1-w}^{-4}$, and

$\frac{\ddot{w}}{zz} + w = \frac{1}{\Sigma} \text{ into } \Sigma - b - \overline{P' \text{cof. } pz + Q' \text{cof. } qz}$ &c. $\times \overline{1-w}^{-3}$

$- \frac{\dot{w}}{z} \times \overline{P \text{fin. } pz + Q \text{fin. } qz}$ &c. $\times \overline{1-w}^{-4}$.

Now, the orbit being supposed nearly circular, we may, in
order to a first approximation, neglect w in both the factors
 $\overline{1-w}^{-3}$ and $\overline{1-w}^{-4}$, as being very small in respect of unity;
by which means Σ will become $= 1 + 2 \text{ flu. } p \dot{z} \text{fin. } pz + Q \dot{z}$
 $\text{fin. } qz$ &c. $= 1 + d - \frac{2P}{p} \times \text{cof. } pz - \frac{2Q}{q} \times \text{cof. } qz$ &c. (see
p. 82.) where d , representing the necessary correction to the flu-
ent, must be taken $= \frac{2P}{p} + \frac{2Q}{q}$ &c. so that Σ may be $= 1$,
when $z=0$. This value of Σ being now substituted in the se-
cond equation $\frac{\ddot{w}}{zz} + w = \frac{1}{\Sigma} \times \Sigma - b - \overline{P' \text{cof. } pz - Q' \text{cof. } qz}$ &c.

(where $\frac{\dot{w}}{z} \times \overline{P \text{fin. } pz + Q \text{fin. } qz}$ &c. on account of the small-

* This assumption is not the less general by the multiples of z being taken the same
here as in the value of Δ ; because, if any multiple of z , in the one value, enters
not into the other, it is but supposing the corresponding coefficient in this last, to van-
ish or become equal to nothing.

nefs of $\frac{\dot{w}}{z}$), there cometh out

$$\frac{\ddot{w}}{zz} + w = \frac{1}{\Sigma} \times 1 + d - b - \frac{2P}{p} + P' \times \text{cof. } pz - \frac{2Q}{q} + Q' \times \text{cof. } qz \&c.$$

where the general multiplier $\frac{1}{\Sigma}$ may be also omitted, as differing very little from unity: this being done, and the fluent being taken, according to the method on *p. 92*, we thence find $w = 1 + d$

$$- b + \alpha \text{ cof. } z - \frac{P' \text{ cof. } pz}{1 - pp} - \frac{Q' \text{ cof. } qz}{1 - qq} \&c. \text{ where } P' = \frac{2P}{p} + P',$$

$$Q' = \frac{2Q}{q} + Q', \&c. \text{ and where the term } \alpha \text{ cof. } z, \text{ by which}$$

the fluent is corrected, must have its coefficient so taken, that w and z may have their origin together, that is, α must be made

$$= b - 1 - d + \frac{P'}{1 - pp} + \frac{Q'}{1 - qq} \&c.$$

Having thus found a value nearly equal to w , we may by help thereof, proceed now to a second approximation, by substituting that value for w , in the factors $\frac{1}{1-w}^{-3}$ and $\frac{1}{1-w}^{-4}$, wherein it was before neglected; and, to facilitate the computation, the terms in the value of $1-w$ (whose reciprocal is the distance of the body from the center of force) may be expressed by the general series of cosines, $e \times 1 - B \text{ cof. } \beta z - C \text{ cof. } \gamma z - D \text{ cof. } \delta z \&c.$ (as it appears from above, that the value of w will consist of such): by which means the same terms before determined will be again brought out, together with a number of others, serving as a farther correction. But, since the former operation is made, more with a view to discover the form of the series, than to be regarded for its exactness, I shall have no farther reference thereto, but proceed to determine the value of the several quantities e , B , C &c. *de novo*, by a method something different from that used above.

First, then, from the equation

$$1 - w = e \times 1 - B \text{ cof. } \beta z - C \text{ cof. } \gamma z - \&c. \text{ will be had}$$

$$\frac{1}{1-w}^{-3} = \frac{1}{e^3} + \frac{3}{e^2} \times B \text{ cof. } \beta z + C \text{ cof. } \gamma z \&c. + \&c.$$

and

and $\overline{1-w}^{-4} = \frac{1}{e^4} + \frac{4}{e^4} \times \overline{B \text{ cof. } \beta z + C \text{ cof. } \gamma z \&c.} + \&c.$

which last value being multiplied by $Pz \times \text{fin. } pz$ (according to the purport of our first equation, $\Sigma = 1 + 2 \text{ fluent } \overline{Pz \text{ fin. } pz + Qz \text{ fin. } qz \&c.} \times \overline{1-w}^{-4}$), and a proper regard being, at the same time, had to the *Theorems on p. 80*, the product will stand thus,

$$\frac{Pz}{e^4} \times \text{fin. } pz + \frac{2Pz}{e^4} \times \overline{B \text{ fin. } \beta - p. z + B \text{ fin. } \beta + p. z \&c.}$$

or thus,

$$\frac{Pz}{e^4} \times \text{fin. } pz + \frac{2Pz}{e^4} \times \overline{B \text{ f. } p-\beta. z + B \text{ f. } p+\beta. z + C \text{ f. } p-\gamma. z + C \text{ f. } p+\gamma. z \&c.}$$

whereof the fluent will be

$$-\frac{P}{e^4} \times \text{cof. } pz - \frac{2P}{e^4} \times \overline{\frac{B \text{ cof. } p-\beta. z}{p-\beta} + \frac{B \text{ cof. } p+\beta. z}{p+\beta} + \frac{C \text{ cof. } p-\gamma. z}{p-\gamma} + \frac{C \text{ cof. } p+\gamma. z}{p+\gamma} \&c.}$$

In the very same manner, the terms arising from the multiplication of $Qz \times \text{fin. } qz$ will be exhibited; and we shall therefore have

$$\Sigma = \left\{ \begin{array}{l} 1+d - \frac{2}{e^4} \times \frac{P}{p} \times \text{cof. } pz + \frac{Q}{q} \times \text{cof. } qz + \frac{R}{r} \times \text{cof. } rz \&c. \\ - \frac{4P}{e^4} \times \overline{\frac{B \text{ cof. } p-\beta. z}{p-\beta} + \frac{B \text{ cof. } p+\beta. z}{p+\beta} + \frac{C \text{ cof. } p-\gamma. z}{p-\gamma} + \frac{C \text{ cof. } p+\gamma. z}{p+\gamma} \&c.} \\ - \frac{4Q}{e^4} \times \overline{\frac{B \text{ cof. } q-\beta. z}{q-\beta} + \frac{B \text{ cof. } q+\beta. z}{q+\beta} + \frac{C \text{ cof. } q-\gamma. z}{q-\gamma} + \frac{C \text{ cof. } q+\gamma. z}{q+\gamma} \&c.} \end{array} \right.$$

In which the quantity d , assumed to denote the necessary correction, must be so taken, that Σ may be $= 1$, when $z=0$,

that is, d must be equal to $\frac{2}{e^4} \times \frac{P}{p} + \frac{Q}{q} + \frac{R}{r} \&c. +$

$$\frac{4P}{e^4} \times \overline{\frac{B}{p-\beta} + \frac{B}{p+\beta} + \frac{C}{p-\gamma} + \frac{C}{p+\gamma} \&c.} +$$

$$\frac{4Q}{e^4} \times \overline{\frac{B}{q-\beta} + \frac{B}{q+\beta} + \frac{C}{q-\gamma} + \frac{C}{q+\gamma} \&c.} + \&c. \text{ Proceeding}$$

now to substitute in our other equation $\left(\frac{w}{z} + w \times \Sigma - \Sigma + b + \overline{P \text{ cof. } pz + Q \text{ cof. } qz \&c.} \times \overline{1+w}^{-3} + \right.$

$\frac{w}{z} \times \overline{P \sin.pz + Q \sin.qz \&c. \times \overline{1-w}^{-4} = 0}$ we shall, in the first place, by taking the fluxion of $w = 1 - e + ex \overline{B \cos.\beta z + C \cos.\gamma z \&c.}$ have $\frac{w}{z} = -e \times \overline{\beta B \sin.\beta z + \gamma C \sin.\gamma z \&c.}$ (*vid. p. 82 and 83.*) whereof the fluxion being, again, taken, we get

$$\frac{w}{z} = -e \times \overline{\beta^2 B \sin.\beta z + \gamma^2 C \sin.\gamma z \&c.} \text{ and consequently}$$

$$\frac{w}{z} + w = 1 - e + ex \overline{1 - \beta\beta \times B \cos.\beta z + 1 - \gamma\gamma \times C \cos.\gamma z \&c.}$$

$$\text{Hence } \frac{w}{z} + w \times \Sigma - \Sigma = -e \Sigma +$$

$$e \times \overline{1 - \beta\beta \times B \cos.\beta z + 1 - \gamma\gamma \times C \cos.\gamma z \&c. \times 1 + d - \frac{2}{e^4} \times \frac{P}{p} \cos.pz + \frac{Q}{q} \cos.qz \&c.}$$

which (by an actual multiplication of terms of the two series's into each other, and a proper application of *Lem. 1, on p. 76*, neglecting at the same time, all terms wherein two, or more dimensions of the quantities *B, C, D* &c. would arise) will be reduced to

$$\begin{aligned} & -e \Sigma + 1 + d. \overline{1 - \beta\beta.e B \cos.\beta z + 1 + d. 1 - \gamma\gamma.e C \cos.\gamma z + 1 + d. 1 - \delta\delta.e D \cos.\delta z \&c.} \\ & - \overline{1 - \beta\beta.} \frac{B}{e^3} \times \frac{P}{p} \times \overline{\cos.p - \beta.z + \cos.p + \beta.z} + \frac{Q}{q} \times \overline{\cos.q - \beta.z + \cos.q + \beta.z \&c.} \\ & - \overline{1 - \gamma\gamma.} \frac{C}{e^3} \times \frac{P}{p} \times \overline{\cos.p - \gamma.z + \cos.p + \gamma.z} + \frac{Q}{q} \times \overline{\cos.q - \gamma.z + \cos.q + \gamma.z \&c.} \\ & \&c. \qquad \&c. \end{aligned}$$

$$\text{In like manner we have } \overline{P' \cos.pz + Q' \cos.qz \&c. \times \overline{1-w}^{-3} = P' \cos.pz + Q' \cos.qz \&c. \times \frac{1}{e^3} + \frac{3}{e^3} \times B \cos.\beta z + C \cos.\gamma z \&c.}$$

$$\begin{aligned} & \left\{ \frac{1}{e^3} \times \overline{P' \cos.pz + Q' \cos.qz + R' \cos.rz \&c.} \right. \\ & + \frac{3P'}{2e^3} \times \overline{B \cos.p - \beta.z + B \cos.p + \beta.z + C \cos.p - \gamma.z + C \cos.p + \gamma.z \&c.} \\ & + \frac{3Q'}{2e^3} \times \overline{B \cos.q - \beta.z + B \cos.q + \beta.z + C \cos.q - \gamma.z + C \cos.q + \gamma.z \&c.} \\ & \left. \&c. \qquad \&c. \right\} \end{aligned}$$

Lastly because $\frac{w}{z} = -e \times \overline{\beta B \sin.\beta z + \gamma C \sin.\gamma z \&c.}$ it fol-

low_s

lows that $\frac{w}{z} \times P \sin. pz + Q \sin. qz \&c. \times \overline{1-w}^{-4} =$
 $-\overline{ex \beta B \sin. \beta z + \gamma C \sin. \gamma z \&c.} \times \overline{P \sin. pz + Q \sin. qz \&c.}$
 $\times \frac{1}{e^3} \times \frac{4}{e^3} \times \overline{B \cos. \beta z + C \cos. \gamma z \&c.}$ whence, by proceeding as a-

bove (neglecting the latter part, $\frac{4}{e^3} \times \overline{B \cos. \beta z + C \cos. \gamma z \&c.}$ of the last factor, as producing terms involving two dimensions of the quantities B, C, D, &c.) will be had

$$-\frac{P}{e^3} \times \overline{\beta B \times \cos. p-\beta. z - \beta B \cos. p+\beta. z + \gamma C \cos. p-\gamma. z - \gamma C \cos. p+\gamma. z \&c.}$$

$$-\frac{Q}{e^3} \times \overline{\beta B \cos. q-\beta. z - \beta B \cos. q+\beta. z + \gamma C \cos. q-\gamma. z - \gamma C \cos. q+\gamma. z \&c.}$$

Now let the three values, thus determined, be collected together, taking instead of Σ , its equal, as given by the first equation, putting $f=1+d$, and dividing the whole by $\overline{1+d.e}$; by which means our equation is, at length, reduced to

$$\overline{1-\beta\beta. B \cos. \beta z + 1-\gamma\gamma. C \cos. \gamma z + 1-\delta\delta. D \cos. \delta z \&c.}$$

$$\left. \begin{aligned} & -1 + \frac{b}{ef} + \frac{1}{e^4 f} \times \frac{2P}{p} + P \times \cos. pz + \frac{2Q}{q} + Q \times \cos. qz \&c. \\ & + \frac{4P}{p-\beta} + \frac{3P'}{2} - \frac{\beta P}{2} - \frac{1-\beta\beta.}{p} \times \frac{B}{e^4 f} \cdot \cos. p-\beta. z \\ & + \frac{4P}{p+\beta} + \frac{3P'}{2} + \frac{\beta P}{2} - \frac{1-\beta\beta.}{p} \times \frac{B}{e^4 f} \cdot \cos. p+\beta. z \\ & + \frac{4P}{p-\gamma} + \frac{3P'}{2} - \frac{\gamma P}{2} - \frac{1-\gamma\gamma.}{p} \times \frac{C}{e^4 f} \cdot \cos. p-\gamma. z \\ & + \frac{4P}{p+\gamma} + \frac{3P'}{2} + \frac{\gamma P}{2} - \frac{1-\gamma\gamma.}{p} \times \frac{C}{e^4 f} \cdot \cos. p+\gamma. z \\ & \&c. \quad \&c. \end{aligned} \right\} = 0$$

But as the coefficients of the terms in this equation are much compounded, it will be proper to make a substitution for them:

Thus, let $\overline{P} = \frac{2}{p} + \frac{P'}{P} \times \frac{P}{e^4 f}$, $\overline{Q} = \frac{2}{q} + \frac{Q'}{Q} \times \frac{Q}{e^4 f} \&c.$

$$\overline{PB1} = \frac{4}{p-\beta} + \frac{3P'}{2P} - \frac{\beta}{2} - \frac{1-\beta\beta.}{p} \times \frac{PB}{e^4 f},$$

$$\overline{PB2} = \frac{4}{p+\beta} + \frac{3P'}{2P} + \frac{\beta}{2} - \frac{1-\beta\beta.}{p} \times \frac{PB}{e^4 f},$$

$$\overline{PC\ 1} = \frac{4}{p-\gamma} + \frac{3P'}{2P} - \frac{\gamma}{2} - \frac{1-\gamma\gamma}{p} \times \frac{PC}{ef},$$

$$\overline{PC\ 2} = \frac{4}{p+\gamma} + \frac{3P'}{2P} + \frac{\gamma}{2} - \frac{1-\beta\beta}{p} \times \frac{PC}{ef}, \&c.$$

supposing these substitutions to be continued on, to take in the terms affected by the other given quantities $Q, q, R, r, \&c.$ (which are had from those above, by barely writing Q and q &c. in the place of P and p &c.) This being done, our equation will stand thus

$$\left. \begin{aligned} &\overline{1-\beta\beta}. B \text{ cof. } \beta z + \overline{1-\gamma\gamma}. C \text{ cof. } \gamma z + \overline{1-\delta\delta}. D \text{ cof. } \delta z \\ &\&c. \\ &-\overline{1} + \frac{b}{ef} + \overline{P} \times \text{cof. } pz + \overline{Q} \times \text{cof. } qz \&c. \\ &+ \overline{PB\ 1} \times \text{cof. } \overline{p-\beta}. z + \overline{PB\ 2} \times \text{cof. } \overline{p+\beta}. z \\ &+ \overline{QB\ 1} \times \text{cof. } \overline{q-\beta}. z + \overline{QB\ 2} \times \text{cof. } \overline{q+\beta}. z \&c. \\ &+ \overline{PC\ 1} \times \text{cof. } \overline{p-\gamma}. z + \overline{PC\ 2} \times \text{cof. } \overline{p+\gamma}. z \\ &+ \overline{QC\ 1} \times \text{cof. } \overline{q-\gamma}. z + \overline{QC\ 2} \times \text{cof. } \overline{q+\gamma}. z \&c. \\ &\&c. \qquad \&c. \end{aligned} \right\} = 0$$

From whence, by comparing the multiples of the arch z , in the first and second lines, we have $\gamma=p$, $\delta=q$, and so on, to as many values (n) as there are quantities $p, q, r, \&c.$ And, by equating the corresponding coefficients of those equi-multiples, we also have $C = -\frac{\overline{P}}{1-p\gamma}$, $D = -\frac{\overline{Q}}{1-q\gamma}$,

$E = -\frac{\overline{R}}{1-r\gamma}$, and so on, to n terms. Then, the value of the first n terms of the series $B \text{ cof. } \beta z + C \text{ cof. } \gamma z + D \text{ cof. } \delta z \&c.$ (exclusive of $B \text{ cof. } \beta z$, of which more hereafter) being thus known, the terms in the 3d, 4th, 5th, &c. lines of our equation, being compounded of them and the given quantities $P, p, Q, q, \&c.$ will also become known; from whence, by continuing the comparison with the terms of the upper line, after those already taken, a new set of terms, involving two dimensions of the quantities $B, C, D, \&c. P, P', Q, Q' \&c.$ will be determined: by means whereof, still continuing the operation in the same manner, a third set of terms may be obtained; and so on, at pleasure.

As to the first term $\overline{1-\beta\beta} \cdot B \cos. \beta z$, whereof no use has been yet made, it is reserved to take off, or destroy any other term, or terms, of the same species, that may arise in the general equation. If no such term should occur, it is but making the coefficient $\overline{1-\beta\beta} \cdot B=0$, and every thing will be right: But that such terms do actually arise, will appear in the following illustration of the general method of proceeding, applied to a particular case, whereon the determination of the lunar orbit depends.

Let $P'=P$, $Q'=\frac{1}{r}P$, and $Q, R \&c.$ all equal to nothing *; then our equation will become

$$\left. \begin{aligned} & \overline{1-\beta\beta} \cdot B \cos. \beta z + \overline{1-\gamma\gamma} \cdot C \cos. \gamma z + \overline{1-\delta\delta} \cdot D \cos. \delta z + \overline{1-\varepsilon\varepsilon} \cdot E \cos. \varepsilon z \&c. \\ & -1 + \frac{b}{ef} + \overline{P} \times \cos. pz + \overline{Q} + \overline{PB_1} \times \cos. \overline{p-\beta} \cdot z + \overline{PB_2} \times \cos. \overline{p+\beta} \cdot z \\ & + \overline{QB_1} + \overline{QB_2} \times \cos. \beta z + \overline{PC_1} \times \cos. \overline{p-\gamma} \cdot z + \overline{PC_2} \times \cos. \overline{p+\gamma} \cdot z \\ & + \overline{QC_1} + \overline{QC_2} \times \cos. \gamma z + \overline{PD_1} \times \cos. \overline{p-\delta} \cdot z + \overline{PD_2} \times \cos. \overline{p+\delta} \cdot z \\ & + \overline{QD_1} + \overline{QD_2} \times \cos. \delta z + \overline{PE_1} \times \cos. \overline{p-\varepsilon} \cdot z + \overline{PE_2} \times \cos. \overline{p+\varepsilon} \cdot z \\ & + \overline{QE_1} + \overline{QE_2} \times \cos. \varepsilon z + \&c. \end{aligned} \right\} = 0$$

Make now, $\gamma=p$, $\delta=p-\beta$, and $\varepsilon=p+\beta$; then the equation will stand thus

$$\left. \begin{aligned} & \overline{1-\beta\beta} \cdot B \cos. \beta z + \overline{1-\gamma\gamma} \cdot C \cos. \gamma z + \overline{1-\delta\delta} \cdot D \cos. \delta z + \&c. \\ & -1 + \frac{b}{ef} + \overline{P} \times \cos. pz + \overline{Q} + \overline{PB_1} \times \cos. \overline{p-\beta} \cdot z + \overline{PB_2} \times \cos. \overline{p+\beta} \cdot z \\ & + \overline{QB_1} + \overline{QB_2} \times \cos. \beta z + \overline{PC_1} + \overline{PC_2} \times \cos. 2pz + \overline{QC_1} + \overline{QC_2} \times \cos. pz \\ & + \overline{PD_1} \times \cos. \beta z + \overline{PD_2} \times \cos. 2\overline{p-\beta} \cdot z + \overline{QD_1} + \overline{QD_2} \times \cos. \overline{p-\beta} \cdot z, \&c. \end{aligned} \right\} = 0$$

Put $B' = \overline{QB_1} + \overline{QB_2} + \overline{PD_1} + \&c.$

$C' = \overline{P} + \overline{QC_1} + \overline{QC_2} + \&c.$

$D' = \overline{PB_1} + \overline{QD_1} + \overline{QD_2} + \&c.$

$E' = \overline{PB_2} + \&c.$

$F' = \overline{PC_2} + \&c.$

$G' = \overline{PD_2} + \&c.$

Then, expunging the terms which have no multiple-cosine in them (as also destroying each other) we, at length have

* That these assumptions are so made, as to express (nearly) the forces whereby the sun disturbs the moon's motion about the earth, will be shewn hereafter.

$$\left. \begin{aligned} & \overline{1-\beta\beta}.B\text{ cof.}\beta z + \overline{1-\gamma\gamma}.C\text{ cof.}\gamma z + \overline{1-\delta\delta}.D\text{ cof.}\delta z + \overline{1-\varepsilon\varepsilon}.E\text{ cof.}\varepsilon z \&c. \\ & + B'\text{ cof.}\beta z + C'\text{ cof.}\beta z + D'\text{ cof.}\beta-\beta.z + E'\text{ cof.}\beta+\beta.z \\ & + F'\text{ cof.}2\beta z + G'\text{ cof.}2\beta-\beta.z + H'\text{ cof.}2\beta+\beta.z + \&c. \end{aligned} \right\} = 0$$

$$\text{whence } \overline{1-\beta\beta} \times B = -B', C = -\frac{C'}{1-\beta\beta}, D = -\frac{D'}{1-\beta-\beta},$$

$$E = -\frac{E'}{1-\beta+\beta}, F = -\frac{F'}{1-4\beta\beta}, G = -\frac{G'}{1-2\beta-\beta}, H = -$$

$$\frac{H'}{1-2\beta+\beta}, \&c. \text{ and consequently}$$

$$\begin{aligned} & \overline{1-w} (= e \times \overline{1-B\text{ cof.}\beta z - C\text{ cof.}\gamma z \&c.}) \\ & = e \times \left\{ \overline{1-B\text{ cof.}\beta z + \frac{C' \times \text{cof.}\beta z}{1-\beta\beta} + \frac{D'\text{ cof.}\beta-\beta.z}{1-\beta-\beta} + \frac{E'\text{ cof.}\beta+\beta.z}{1-\beta+\beta}} \right. \\ & \left. + \frac{F'\text{ cof.}2\beta z}{1-4\beta\beta} + \frac{G'\text{ cof.}2\beta-\beta.z}{1-2\beta-\beta} + \frac{H'\text{ cof.}2\beta+\beta.z}{1-2\beta+\beta} + \&c. \right\} \end{aligned}$$

In deriving the equation here brought out, all terms involving two, or more dimensions of the quantities $B, C, D, \&c.$ are neglected; it will, therefore, be necessary to shew now, how the effect of those terms may be computed, and the approximation carried on, to any farther degree of exactness desired. Previous to which, it will be proper to observe, that, in the preceding calculations, the quantity $\overline{1-w}^{-3}$ was taken =

$$\frac{1}{e^3} + \frac{3}{e^3} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.} \text{ barely, and } \overline{1-w}^{-4} = \frac{1}{e^4} + \frac{4}{e^4} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.} \text{ whereas the true value}$$

$$\text{of } \overline{1-w}^{-3}, \text{ or } e \times \overline{1-B\text{ cof.}\beta z - C\text{ cof.}\gamma z \&c.}^{-3} \text{ is } = \frac{1}{e^3} + \frac{3}{e^3} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.} + \frac{6}{e^3} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.}^2 + \&c.$$

$$\text{and that of } \overline{1-w}^{-4} = \frac{1}{e^4} + \frac{4}{e^4} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.}$$

$$+ \frac{10}{e^4} \times \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.}^2 \text{ Now } \overline{B\text{ cof.}\beta z + C\text{ cof.}\gamma z \&c.}^2$$

(before neglected) is evidently equal to $B^2 \times \overline{\text{cof.}\beta z}^2 + 2 B C \times \overline{\text{cof.}\beta z} \times \overline{\text{cof.}\gamma z \&c.} = \frac{1}{2} B^2 \times \overline{1 + \text{cof.}2\beta z} + BC \times \overline{\text{cof.}\gamma - \beta.z + \text{cof.}\gamma + \beta.z} + \&c.$ all which cosines, together with their coefficients, will be known, seeing β, B, γ, C &c.

&c. are given (or nearly so) by the preceding operation. And, in the same manner, a series of cosines expressing the value of $B \cos. \beta z + C \cos. \gamma z \&c.$ (or of the most considerable terms thereof) may be found, should it be necessary to continue the approximation so far.

To find what alteration these quantities will produce in the value of $1-w$ (before found) let any new term thus arising in the value of $1-w$, be denoted by $\frac{M}{e^3} \times \cos. \alpha z$, and the term

answering to it in the value of $1-w$, by $\frac{N}{e^4} \times \cos. \alpha z$: then the corresponding increase in the value of $-e \Sigma (= -e - 2e \times \text{flu. } P \dot{z} \sin. pz + Q \dot{z} \sin. qz. \&c. \times 1-w)^{-4}$, will be $-2e \times \text{flu. } P \dot{z} \sin. pz + Q \dot{z} \sin. qz. \&c. \times \frac{N}{e^4} \times \cos. \alpha z = -\frac{N}{e^3}$ into flu. $P \dot{z} \times \sin. p - \alpha. z + \sin. p + \alpha. z + Q \dot{z} \times \sin. q - \alpha. z + \sin. q + \alpha. z \&c. = \frac{N}{e^3}$ into $\frac{P \cos. p - \alpha. z}{p - \alpha} + \frac{P \cos. p + \alpha. z}{p + \alpha} + \frac{Q \cos. q - \alpha. z}{q - \alpha} + \frac{Q \cos. q + \alpha. z}{q + \alpha} \&c.$

In like manner, the increase of the second member ($P' \cos. pz + Q' \cos. qz \&c. \times 1-w$) of our general equation (see p. 147.) appears to be $P' \cos. pz + Q' \cos. qz \&c. \times \frac{M}{e^3} \times \cos. \alpha z$

$= \frac{M}{2e^3} \times P' \cos. p - \alpha. z + P' \cos. p + \alpha. z + Q' \cos. q - \alpha. z + Q' \cos. q + \alpha. z \&c.$

And the increase of the last term, or member,

$P \sin. pz + Q \sin. qz \&c. \times \frac{1}{3z} \times \text{flux. } 1-w$, is had =

$P \sin. pz + Q \sin. qz \&c. \times \frac{1}{3z} \times -\frac{M \alpha z}{e^3} \times \sin. \alpha z$

$= P \sin. pz + Q \sin. qz \&c. \times -\frac{M \alpha}{3e^3} \times \sin. \alpha z =$

$-\frac{\alpha M}{6e^3} \times P \cos. p - \alpha. z - P \cos. p + \alpha. z + Q \cos. q - \alpha. z - Q \cos. q + \alpha. z \&c.$

Let these three quantities be now collected together; from whence the new terms entering into the general equation, when the whole is divided by e^3 , will stand thus

$$\frac{P}{e^2 f} \times \frac{N}{p-a} + \frac{MP}{2P} - \frac{aM}{6} \times \text{cof. } p-a.z + \frac{P}{e^2 f} \times \frac{N}{p+a} + \frac{MP}{2P} + \frac{aM}{6} \times \text{cof. } p+a.z + \&c.$$

So that, from the general method of operation before laid down, it appears that the proper correction for the value of $1-w$, arising from the terms, $\frac{M}{e^2} \times \text{cof. } az$ and $\frac{N}{e^2} \times \text{cof. } az$, before neglected in the respective values of $1-w$ ⁻³ and $1-w$ ⁻⁴,

will be expressed by e into $\frac{P}{e^2 f} \times \frac{N}{p-a} + \frac{MP}{2P} - \frac{aM}{6} \times \frac{\text{cof. } p-a.z}{1-p-a_1^2} +$

$$\frac{P}{e^2 f} \times \frac{N}{p-a} + \frac{MP}{2P} + \frac{aM}{6} \times \frac{\text{cof. } p+a.z}{1-p+a_1^2} +$$

$$\frac{Q}{e^2 f} \times \frac{N}{q-a} + \frac{MQ}{2Q} - \frac{aM}{6} \times \frac{\text{cof. } q-a.z}{1-q-a_1^2} +$$

$$\frac{Q}{e^2 f} \times \frac{N}{q+a} + \frac{MQ}{2Q} + \frac{aM}{6} \times \frac{\text{cof. } q+a.z}{1-q+a_1^2} + \&c.$$

In respect to which it may be observed, that no regard has been had to such quantities as would arise from the multiplication of the new terms in Σ , by the series $1-\beta\beta$. $B \text{ cof. } \beta z + 1-\gamma\gamma$. $C \text{ cof. } \gamma z + \&c.$ (as ought, in strictness, to have been done); because these terms being very small, they will, after multiplication into the small quantities $1-\beta\beta \times B$, $1-\gamma\gamma \times C$ &c. be so far reduced, as to become quite inconsiderable. And it may be observed farther, that even the whole product arising from the multiplication of the value of Σ into the said series, except the terms $-e\Sigma + 1-\beta\beta.e f B \text{ cof. } \beta z + 1-\gamma\gamma.e f C \text{ cof. } \gamma z$ &c. might be also rejected, without producing an error of more than a few seconds, in finding the place of the moon in her orbit.

Having shewn what alteration will be caused in the value of $1-w$ (the reciprocal of the distance of the body from the center of force) from the taking in of any small, assigned terms in the values of $1-w$ ⁻³ and $1-w$ ⁻⁴, it may be a proper place here to shew (as it is of great importance to be known) what change will arise in the said value of $1-w$ from the addition of any small, new terms, $A \sin. \pi z$,
and

and $A' \cos. \pi z$, to the respective forces $P \sin. pz + Q \sin. qz$ &c. and $P' \cos. pz + Q' \cos. qz$ &c. whereby the motion of the body is disturbed. What this alteration, or correction ought to be, is easily discovered from the general equation on p. 151; from whence, by substituting π , A , and A' instead of p , P , and P' , respectively, the new terms affected by π , entering into the said equation, will appear to be

$$\frac{A}{e^4 f} \times \frac{2}{\pi} + \frac{A'}{A} \times \cos. \pi z$$

$$+ \frac{4}{\pi - \beta} + \frac{3A'}{2A} - \frac{\beta}{2} - \frac{1 - \beta\beta}{\pi} \times \frac{AB}{e^4 f} \times \cos. \pi - \beta. z$$

$$+ \frac{4}{\pi + \beta} + \frac{3A'}{2A} + \frac{\beta}{2} - \frac{1 - \beta\beta}{\pi} \times \frac{AB}{e^4 f} \times \cos. \pi + \beta. z. \&c.$$

and consequently the increase in the value of $1 - w$ arising

$$\text{therefrom} = e \text{ into } \frac{A}{e^4 f} \times \frac{2}{\pi} + \frac{A'}{A} \times \frac{\cos. \pi z}{1 - \pi\pi} +$$

$$\frac{4}{\pi - \beta} + \frac{3A'}{2A} - \frac{\beta}{2} - \frac{1 - \beta\beta}{\pi} \times \frac{AB}{e^4 f} \times \frac{\cos. \pi - \beta. z}{1 - \pi\pi} +$$

$$\frac{4}{\pi + \beta} + \frac{3A'}{2A} + \frac{\beta}{2} - \frac{1 - \beta\beta}{\pi} \times \frac{AB}{e^4 f} \times \frac{\cos. \pi + \beta. z}{1 - \pi\pi} + \&c.$$

where, in many cases, the first term alone, will be sufficient.

In making the different corrections above pointed out, it will be found necessary to have a particular regard to such multiples of the arch z , as are, either, very small, or nearly equal to unity (of which two kinds, those whose exponents are $p - 2\beta$, and $p - \beta$, will be found the most considerable.) For, though the coefficients of such terms should appear, at first, to be small, they ought not, therefore, to be immediately rejected; because the divisors which they afterwards receive (the former in obtaining the value of $1 - w$, and the latter, in finding the anomaly from thence) are such as may render the effect, or quotient, too considerable to be intirely disregarded.

And it may be easily conceived, without the help of calculation, that a term, or force of the former kind, expressed by the sine or co-sine of a very small multiple of the longitude z , must necessarily have a much greater effect, than another (having the same coefficient) which is proportional to the sine or co-sine of a large multiple of the same angle: because, when the

the *index*, or *multiple* is a very small one, the term itself, while z increases, will continue, for a considerable time, nearly of the same value; and consequently, will have its whole effect exerted in the same direction; but when the *multiple* is a large one, the changes from *positive* to *negative*, and from thence to *positive* again, are so quick, that sufficient time is not allowed for producing any considerable inequality in the body's motion, before that effect is again destroyed by the same force, acting equally in the opposite direction.

The value of $1-w$ (the reciprocal of the body's distance from the center of force) being, by the *formulæ* laid down above, approximated to a sufficient degree of exactness, we may from thence, and the equation $\dot{t} = \frac{g}{2c} \times \frac{z}{\Sigma^{\frac{1}{2}} \times (1-w)^2}$ (given at page 146.) proceed to compute the time (t) of describing the angle z ; whereby the difference between the *true* and *mean* anomalies will also be known; which, in the *lunar theory*, is the great point in question, and is besides, absolutely necessary in order to introduce the proper quantities of the forces whereby the moon's motion is disturbed.

Let, therefore, the value of Σ (as given by the first equation, at p. 149,) be here represented by

$f \times 1 - \overline{B} \cos. \beta z - C \cos. \gamma z - \overline{D} \cos. \delta z \&c.$ so shall

$$\dot{t} = \frac{g}{2c} \times f^{\frac{1}{2}} \times [1 - \overline{B} \cos. \beta z - C \cos. \gamma z \&c.]^{-\frac{1}{2}} \times e^{-2} \times [1 - B \cos. \beta z \&c.]^{-2}$$

$$\text{But } [1 - \overline{B} \cos. \beta z - C \cos. \gamma z \&c.]^{-\frac{1}{2}} = 1 + \frac{1}{2} \times \overline{B} \cos. \beta z \&c. + \frac{3}{8} \times \overline{B}^2 \cos. \beta z + \overline{C} \cos. \gamma z \&c.]^{\frac{1}{2}} = 1 + \frac{1}{2} \times \overline{B} \cos. \beta z + \overline{C} \cos. \gamma z \&c. +$$

$$\frac{1}{2} \times \frac{1}{2} \overline{B} \overline{B} \times 1 + \cos. 2 \beta z + B C \times \cos. \gamma - \beta. z + \cos. \gamma + \beta. z \&c.$$

$$\text{And } [1 - \overline{B} \cos. \beta z - C \cos. \gamma z \&c.]^{-2} = 1 + 2 \times \overline{B} \cos. \beta z \&c. +$$

$$3 \times \frac{1}{2} \overline{B} \overline{B} \times 1 + \cos. 2 \beta z + B C \times \cos. \gamma - \beta. z + \cos. \gamma + \beta. z \&c.$$

Which values being multiplied together, we thence get $\dot{t} =$

$$\frac{g z}{2 c e f^{\frac{1}{2}}} \text{ into } 1 + \left\{ 2 \times \overline{B} \cos. \beta z + C \cos. \gamma z \&c. \right. \\ \left. + \frac{1}{2} \times \overline{B} \cos. \beta z + \overline{C} \cos. \gamma z \&c. \right.$$

+

$$+ \left\{ \begin{array}{l} 3 \times \frac{1}{2} BB \times I + \text{cof. } 2\beta z + BC \times \text{cof. } \gamma - \beta. z + \text{cof. } \gamma + \beta. z \&c. \\ + \frac{3}{8} \times \frac{1}{2} \bar{B}\bar{B} \times I + \text{cof. } 2\beta z + \bar{B}\bar{C} \times \text{cof. } \gamma - \beta. z + \text{cof. } \gamma + \beta. z \&c. \\ + \frac{1}{2} \times BB \times I + \text{cof. } 2\beta z + \bar{B}\bar{C} + \bar{B}\bar{C} \times \text{cof. } \gamma - \beta. z + c. \gamma + \beta. z \&c. \end{array} \right.$$

Put $b = I + \frac{1}{2} \times BB + CC + DD \&c. + \frac{1}{10} \times BB + \bar{C}\bar{C} + \bar{D}\bar{D} \&c.$
 $+ \frac{1}{2} \times BB + \bar{C}\bar{C} + \bar{D}\bar{D} \&c. (B) = 2B + \frac{1}{2} \bar{B}, (BB) = \frac{1}{2} BB + \frac{1}{8} \bar{B}\bar{B} + \frac{1}{2} B\bar{B}, (BC) = 3BC + \frac{1}{8} \bar{B}\bar{C} + \frac{1}{2} \bar{B}C + \frac{1}{2} B\bar{C}, \&c.$ continuing on these substitutions, so as to take in the terms affected by the other quantities D, E, F, &c. (which are had from those above, by barely writing one letter for another); by means whereof our equation is reduced to

$$t = \frac{gz}{2c^2 f^{\frac{1}{2}}} \times \left\{ \begin{array}{l} b + (B) \times \text{cof. } \beta z + (C) \times \text{cof. } \gamma z + (D) \times \text{cof. } \delta z \&c. \\ (BB) \times \text{cof. } 2\beta z + (BC) \times \text{cof. } \gamma - \beta. z + \text{cof. } \gamma + \beta. z \\ (CC) \times \text{cof. } 2\gamma z + (CD) \times \text{cof. } \gamma - \delta. z + \text{cof. } \gamma + \delta. z \\ \&c. \end{array} \right.$$

From whence, by taking the fluent &c. we have

$$t = \frac{gb}{2c^2 f^{\frac{1}{2}}} \times \left\{ \begin{array}{l} z + \frac{(B) \times \text{fin. } \beta z}{b\beta} + \frac{(C) \times \text{fin. } \gamma z}{b\gamma} + \frac{(D) \times \text{fin. } \delta z}{b\delta} + \&c. \\ \frac{(BB) \times \text{fin. } 2\beta z}{2b\beta} + \frac{(BC)}{b} \times \frac{\text{fin. } \gamma - \beta. z}{\gamma - \beta} + \frac{\text{fin. } \gamma + \beta. z}{\gamma + \beta} + \&c. \\ \frac{(CC) \times \text{fin. } 2\gamma z}{2b\gamma} + \frac{(CD)}{b} \times \frac{\text{fin. } \gamma - \delta. z}{\gamma - \delta} + \frac{\text{fin. } \gamma + \delta. z}{\gamma + \delta} + \&c. \end{array} \right.$$

In respect to which it will be needful to observe, that the first term, $\frac{gb}{2c^2 f^{\frac{1}{2}}} \times z$, will, when $z =$ the whole circumference,

be the true measure of the mean periodic time; because all the other terms being composed of the sines of multiples of the arch z , they will, while z keeps increasing, change from *positive* to *negative*, and from *thence* to *positive* again, and so on continually; and therefore can have nothing to do in the *mean motion*; being themselves no other than the proper equations whereby the *mean* and *true* motions differ from each other; so that, the *true motion* being defined by z , the *mean motion* will be expressed by $z + \frac{(B) \text{fin. } \beta z}{b\beta} + \frac{(C) \text{fin. } \gamma z}{b\gamma} \&c.$ as above determined.

From

Fig. 31. From the expreffion $\frac{gb}{2c^2 f^{\frac{1}{2}}} \times z$, here found, the proportion between the mean periodic time of the body in its orbit ABP &c. and the periodic time in the circular orbit ADK, that might be described, independent of the perturbing forces, by means of a centripetal force fufficient to caufe the body to move therein, will be known: for the quantities c , f , and b being here, each, equal to unity, the faid expreffion will, in this cafe, become $= \frac{g}{2c} \times z$: whence it is evident, that the periodic time in the circle, will be in proportion to the periodic time in the orbit ABP &c. as *unity* is to $\frac{b}{c^2 f^{\frac{1}{2}}}$.

APPLICATION to the *Lunar Orbit*.

In order to apply the conclufions derived in the preceding pages, to the determination of the lunar orbit and the different *inequalities* of the motion therein, it will be neceffary, firft of all, to investigate the fun's force to difturb the motion of the moon about the center of the earth; from whence all thofe *inequalities*, except *that* arifing from the excentricity, are produced.

Fig. 34. Let C, S, and B represent any three cotemporary places of the earth, fun, and moon, refpectively; and, upon the diagonal BS, let the parallelogram BCSH be conftituted; making BF perpendicular to CS.

If k be affumed to denote accelerative force of the earth to the fun, the accelerative force of the moon to the fun will be truly represented by $k \times \frac{\overline{SC}^2}{\overline{SB}^3}$; which force may be refolved into two others, the one in the direction BC, expreffed by $k \times \frac{\overline{SC}^2}{\overline{SB}^3} \times \frac{BC}{SC}$; and the other in the direction BH, expreffed by $k \times \frac{\overline{SC}^2}{\overline{SB}^3} \times \frac{SC}{SB}$; from which laft, let the force k in the parallel direction CS be fubtracted; fo fhall the remainder

$k \times$

$k \times \frac{\overline{SC} - \overline{SB}^3}{\overline{SB}^3}$ be that part of the force, acting in the direction BH, whereby the motion of the moon about the earth is disturbed: But this quantity, $k \times \frac{\overline{SC} - \overline{SB}^3}{\overline{SB}^3}$, is evidently equal to $k \times \overline{SC} - \overline{SB} \times \frac{\overline{SC}^2 + \overline{SC} \times \overline{SB} + \overline{SB}^2}{\overline{SB}^3}$; which, as SB (by reason of the great distance of the sun) is nearly equal to SF, will be also equal to $k \times \overline{CF} \times \frac{\overline{SC}^2 + \overline{SC} \times \overline{SB} + \overline{SB}^2}{\overline{SB}^3}$, or to $3k \times \frac{\overline{CF}}{\overline{SC}}$, very near; because SC being nearly equal to SB, it may be here substituted instead thereof. Now this force $3k \times \frac{\overline{CF}}{\overline{SC}}$, acting in the direction BH, parallel to SC, may be, again, resolved into two others; the one in a direction perpendicular to BC, expressed by $3k \times \frac{\overline{CF}}{\overline{SC}} \times \sin. SCB = 3k \times \frac{\overline{CB}}{\overline{SC}} \times \sin. SCB \times \cos. SCB = \frac{3k}{2} \times \frac{\overline{CB}}{\overline{SC}} \times \sin. 2SCB$; and the other in the direction of CB, expressed by $3K \times \frac{\overline{CF}}{\overline{SC}} \times \cos. SCB = 3k \times \frac{\overline{CB}}{\overline{SC}} \times \cos. SCB = \frac{3k}{2} \times \frac{\overline{CB}}{\overline{SC}} \times 1 + \cos. 2SCB$: from which last, let the force $k \times \frac{\overline{SC}^2}{\overline{SB}^3} \times \frac{\overline{BC}}{\overline{SB}}$ (or $k \times \frac{\overline{BC}}{\overline{SC}}$) above found, acting in the opposite direction BC, be subtracted, so shall the remainder $k \times \frac{\overline{CB}}{\overline{SC}} \times \frac{1}{2} + \frac{1}{2} \cos. 2SCB$ be the diminution of the centripetal force to the earth, arising from the action of the sun.

To exterminate k and CS from these expressions, let the given quantity 0.0748, expressing the mean periodic time of the moon, in parts of an year (*as found by observation*) be denoted by m ; then (*by the proportion on p. 160.*) as $\frac{h}{e^2 f^2}$ is to 1, so is m to $\left(\frac{m e^2 f^2}{h} \right)$ the periodic time in the circular orbit ADK, that might be described, independent of the perturbing forces, by means of a centripetal force sufficient to cause the moon to revolve therein. But in circles the centripetal forces are

Y known

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known to be as the radii directly, and the squares of the periodic times inversely; whence we have, as 1 (the force in the circle ADK) is to k , the mean force whereby the earth is retained

in its orbit about the sun, so is $\frac{CA}{\frac{me^2f^2}{bh}}$ to $\frac{CS}{1^2}$: from which proportion

$\frac{k}{CS} = \frac{m^2e^2f^2}{bh \times CA}$: And the forces $k \times \frac{CB}{CS} \times \frac{1}{\frac{1}{2} + \frac{1}{2} \cos. 2SCB}$,

and $\frac{k \times CB}{CS} \times \frac{1}{2} \sin. 2SCB$, whereby the motion of the moon about the earth is disturbed, will therefore be truly defined by $\frac{3m^2e^2f^2}{2bh} \times \frac{CB}{CA} \times \cos. 2SCB + \frac{1}{2}$, and $\frac{3m^2e^2f^2}{2bh} \times \frac{CB}{CA} \times \sin. 2SCB$.

If the sun and moon be supposed to move from the line CA at the same time, so that the angle ACS may be the sun's apparent motion about the earth, whilst the moon in her orbit moves from A to B, it will be, as $1:m::z(= \text{the angle ACB}):mz = \text{the angle ACS}$, *nearly*; which would be strictly true, were the *true motions* of the sun and moon to be exactly in the same proportion with the *mean motions*. Hence SCB (= ACB — ACS) will be had = $z - mz$, *nearly*; and consequently $2SCB = pz$ (by making $p = 1 - m \times 2$): and so the forces found above, by substituting this value, and writing $\frac{1}{1-w}$ in the

room of its equal $\frac{CB}{CA}$, will, by means thereof, be reduced to

$\frac{1}{1-w} \times \frac{3m^2e^2f^2}{2bh} \times \cos. pz + \frac{m^2e^2f^2}{2bh}$, and $\frac{1}{1-w} \times \frac{3m^2e^2f^2}{2bh} \times \sin. pz$: which quantities, with contrary signs (because they diminish the centripetal force to the earth, and the area described, instead of increasing them) being compared with the two general expressions $\frac{1}{1-w} \times P' \cos. pz + Q' \cos. qz + R' \cos. rz$ &c. and

$\frac{1}{1-w} \times P \sin. pz + Q \sin. qz + R \sin. rz$ &c. as given on p. 147,

* In deriving this conclusion, regard is had to the moon's relative motion about the earth's center; but if we consider the motion, as performed about the common center of gravity of the earth and moon, the result will be exactly the same.

we shall, in this case, have $q = 0$, $r = 0$, &c. $P' = -\frac{3m^2}{2} \times \frac{e^4 f}{bb} = -0,0084 \times \frac{e^4 f}{bb}$, $Q = -\frac{m^2}{2} \times \frac{e^4 f}{bb} = -0,0028 \times \frac{e^4 f}{bb}$ ($= \frac{1}{3} P'$), $P = -\frac{3m^2}{2} \times \frac{e^4 f}{bb}$ ($= P'$), $Q = 0$, $R = 0$ &c. But here, instead of b , or its equal, $1 + \frac{1}{2} \times \overline{BB+CC+DD}$ &c. (given at *p.* 159,) it will be sufficient to make use of the first term of the series only (till a more exact value, by means of the subsequent calculations, can be known); because all the quantities B , C , D , &c. being small in comparison of *unity*, the squares of them, which are here neglected, will be still smaller, and of less consequence.

If, now, the case under consideration be compared with *that* laid down, and resolved, at *p.* 153, they will appear to be the same; so that we have nothing more to do here, than to compute, in numbers, the different values of the algebraic quantities there brought out. But one thing previous thereto must be taken notice of, respecting the principal ($B \cos. \beta z$) of those values, on which the great elliptical equation, arising from the eccentricity, depends; which cannot be known but from the observations of *Astronomers*; since it is owing to the projectile-velocity which the moon, *first*, received, more than to the perturbing force of the sun, whose effect we are about to calculate. But, though the term $B \times \cos. \beta z$, cannot be determined by *theory* alone, yet the value of its exponent β , on which the motion of the *apogee* depends, may from hence be deduced. This value, in a former operation (see *page* 148,) was found to be an *unit* (as the circumstance of the problem, when the *apogee* is at rest, absolutely requires); the true value cannot, therefore, differ much from an *unit*, which may be used instead of β , till a more exact value, by means of a second operation, is known. Making, therefore, $\beta = 1$; $\frac{P}{e^4 f} = -0,0084$; $P' = P$; $Q = 0$; $Q' = \frac{1}{3} P' = \frac{1}{3} P$; $m = 0,0748$, p ($= 1 - m \times 2$) $= 1,8504$ $\gamma = p$, $\delta = p - \beta$, $\epsilon = p + \beta$, &c. (as before determined) and substituting in the equations on *p.* 151, 152, 153, we have

$$\bar{P} (= \frac{2P}{pe^{4f}} + \frac{P'}{e^{4f}}) = \frac{2x-0,0084}{p} - 0,084 = -0,001748;$$

$$\bar{Q} (= \frac{2Q}{qe^{4f}} + \frac{Q'}{e^{4f}}) = -\frac{0,0084}{3} = -0,0028;$$

$$\bar{PB}_1 (= \frac{4}{p-\beta} + \frac{1}{2} - \frac{\beta}{2} - \frac{1-\beta\beta}{p} \times \frac{PB}{e^{4f}}) = -0,048B;$$

$$\bar{PB}_2 (= \frac{4}{p+\beta} + \frac{1}{2} + \frac{\beta}{2} - \frac{1-\beta\beta}{p} \times \frac{PB}{e^{4f}}) = -0,0285B;$$

$$\bar{PC}_2 (= \frac{4}{p+\gamma} + \frac{1}{2} + \frac{\gamma}{2} - \frac{1-\gamma\gamma}{p} \times \frac{PC}{e^{4f}}) = -0,0404C;$$

$$\bar{PD}_1 (= \frac{4}{p-\delta} + \frac{1}{2} - \frac{\delta}{2} - \frac{1-\delta\delta}{p} \times \frac{PD}{e^{4f}}) = -0,0414D;$$

$$\bar{PD}_2 (= \frac{4}{p+\delta} + \frac{1}{2} + \frac{\delta}{2} - \frac{1-\delta\delta}{p} \times \frac{PD}{e^{4f}}) = -0,0273D;$$

$$\bar{QB}_1 (= \frac{4Q}{q-\beta} + \frac{3Q'}{2} - \frac{EQ}{2} - \frac{1-\beta\beta}{1-\beta\beta} \times \frac{Q}{q} \times \frac{B}{e^{4f}}) = \frac{PB}{2e^{4f}} = -0,0042B \text{ (because } Q=0, \text{ and } Q'=\frac{1}{3}P); \text{ and in the very same manner, } \bar{QB}_2 = -0,0042B; \bar{QC}_1 = -0,0042C; \bar{QC}_2 = -0,0042C; \bar{QD}_1 = -0,0042D, \bar{QD}_2 = -0,0042D, \&c. \text{ *}$$

Whence

* In these calculations, all terms whose divisors are found equal to nothing, will themselves be nothing, and not infinite, as might at first be imagined: thus the term $\frac{\cos p-\gamma z}{p-\gamma}$, by having its divisor $p-\gamma=0$, intirely vanishes. The reason whereof will appear evident, if it be considered, that this

term, $\frac{\cos p-\gamma z}{p-\gamma}$, arises by taking the fluent of $-z \times \sin p-\gamma z$, or $-z \times 0$: which is, manifestly, equal to nothing. But an objection may, perhaps, be brought from hence, against the truth and universality of the *fluxionary calculus*; seeing thereby an expression is here derived, which, though actually equal to nothing, appears nevertheless under the form of an infinite quantity.

To clear up this point, it must be observed, that $\frac{\cos p-\gamma z}{p-\gamma}$ is not the complete fluent of $-z \times \sin p-\gamma z$, but the variable part of it only; the corrected fluent being $-\frac{1}{p-\gamma} + \frac{\cos p-\gamma z}{p-\gamma}$, or $\frac{1}{p-\gamma} \times -1 + \cos p-\gamma z$. But, whatever value $p-\gamma$ is supposed to have, the cosine of $p-\gamma z$ will, it is known, be expressed

Whence it is evident, that

$$\begin{aligned} B' &= (\overline{QB_1} + \overline{QB_2} + \overline{PD_1} \&c.) = -0,0084B - 0,0414D, \\ C' &= (\overline{P} + \overline{QC_1} + \overline{QC_2} \&c.) = -0,01748 - 0,0084C, \\ D' &= (\overline{PB_1} + \overline{QD_1} + \overline{QD_2} \&c.) = -0,048B - 0,0084D, \\ E' &= (\overline{PB_2} \&c.) = -0,0285B, \\ F' &= (\overline{PC_2} \&c.) = -0,0404C, \\ G' &= (\overline{PD_2} \&c.) = -0,0273D, \&c. \end{aligned}$$

But B' being $= -1-\beta\beta.B$, $C' = -1-\gamma\gamma.C$, $D' = -1-\delta\delta.D$, &c. the second and third of these equations, will by substitution, be changed to $1-\gamma\gamma.C = 0,01748 + 0,0084C$, and $1-\delta\delta.D = 0,048B + 0,0084D$; whence C is given $= \frac{0,01748}{1-\gamma\gamma-0,0084} = \frac{0,01748}{1-1,85041^2-0,0084} = -0,007186$; and $D = \frac{0,048B}{1-0,85041^2-0,0084} = 0,178B$: which values being substituted in the other equations, they will become $B' = -0,0158B$, $E' = -0,0285B$, $F' = 0,0003$, $G' = -0,005B$; whence $E (= -\frac{E'}{1-\beta+\beta^2}) = -0,004B$, $F (= -\frac{F'}{1-4\beta\beta}) = 0,000023$, $G (= -\frac{G'}{1-2\beta-\beta^2}) = -0,0008B$; and consequently $1-w = (ex 1-B\cos.\beta z - C\cos.\gamma z - D\cos.\delta z \&c.) = e$ into $1-B\cos.\beta z + 0,007186\cos.\beta z - 0,178B\cos.\beta z + 0,004B\cos.\beta z - 0,000023\cos.2\beta z + 0,0008B\cos.2\beta z \&c.$ expressing the reciprocal of the moon's distance from the earth.

As to the quantity B , which enters into the greater part of the terms of the series here found, it cannot, as has been al-

pressed by $1 - \frac{\overline{p-q}^2 z^2}{1.2} + \frac{\overline{p-\gamma}^4 z^4}{1.2.3.4} \&c.$ So that, the fluent in question will be truly represented by $\frac{1}{p-\gamma} \times -\frac{\overline{p-\gamma}^2 z^2}{1.2} + \frac{\overline{p-\gamma}^4 z^4}{1.2.3.4} \&c.$ or its equal $-\frac{\overline{p-\gamma}^2 z^2}{1.2} + \frac{\overline{p-\gamma}^4 z^4}{1.2.3.4} \&c.$ which entirely vanishes, as it ought to do, when $p-\gamma=0$.

ready

ready intimated, be otherwise determined than from the observations of *Astronomers*: nor will the equation above expressing the relation of B' and B , afford us the least help therein: For, by substituting $-0,0158B$ instead of its equal B' , that equation will become $-0,0158B = -\frac{1-\beta\beta.B}{1-\beta\beta}$, or $1-\beta\beta=0,0158$; where, B intirely vanishing, nothing in relation to it can, therefore, be determined. We have here, indeed, an equation for finding the value of β ; which from thence is given $=\sqrt{1-0,0158}=0,99206$; by means whereof the motion of the *apogee* will be known: for it will appear, by *cor. 1 and 3 to prob. VI.*) that $1-w=e \times 1-B \cos.\beta z$ is the equation corresponding to a moveable ellipse, turning about the focus, or center of force, with an angular celerity which is to that of the body in the ellipse, every-where in the constant proportion of $1-\beta$ to β : whence it follows, that the mean motion of the *apogee*, ought to be in proportion to the mean motion of the moon, as $0,00794$ to *unity*: which differs from the *real* proportion (of $0,008455$ to 1) as given from *observation*, by about $\frac{1}{18}$ part of the whole value: nor ought this to seem strange, as a number of (small) terms yet remain to be introduced into the value of $1-w$, by the corrections pointed out on *p. 156* and *157*: besides which, the difference arising in the co-efficients of the terms already found, by substituting this, *new*, value for β , will amount to something considerable, from whence, alone, near half the error would be taken away. But, to avoid the trouble of repeating the same operation, again and again, with the new values of β , thus found, I shall here, at once, take β equal to the true value ($0,991545$) as given from *observation*; but

If the eccentricity B , be supposed to vanish, $1-w$ will then become $=e \times 1+0,007186 \cos.pz-0,000023 \cos.2pz$ &c. which, when $pz=0$, or the moon is the *syzygy*, will be $=e \times 1+0,007186-0,000023=1,007163 \times e$; but when $\frac{1}{2}pz=90$ degrees, or the moon is in the *quadrature*, it will then be $=e \times 1-0,007186-0,000023=0,992791 \times e$. Therefore, when the orbit is without eccentricity, the distance of the moon from the earth, in the *syzygy* is in proportion to the distance in the *quadrature*, as $\frac{1}{1,007163 \times e}$ to $\frac{1}{0,992791 \times e}$, or as $0,992791$ to $1,007163$, that is, as 69 to 70 , very near; the same as is found by Sir Isaac Newton in his *Princip. B. III. prop. 28.*

shall,

shall, at the same time, put down the several terms arising in the equation for the value of β ; by means whereof it will appear, in case both sides are found to be equal, that the value of the root β has been rightly assumed; and that the motion of the moon's *apogee* (which has been the subject of so much speculation and controversy) is intirely consistent with the general laws of gravitation.

Now, of the quantities above determined, $\overline{PB_1}$ and $\overline{PD_1}$ are those that are the most affected by altering the value of β : these being, therefore, computed a-new (making β equal to 0,991545, instead of unity) the former will here come out $= -0,04747 B$, and the latter $= -0,04168 D$: from whence, by proceeding as before, we have $D=0,1869 B$, and $1-\beta\beta (= -\frac{B'}{B}) = 0,01619$. As to the values of $E, F, G, \&c.$ they are so small, in themselves, and so little affected by β , that to compute them a-new, would be quite unnecessary; the difference not producing an error of a single *second*, in the place of the moon.

To apply, now, the observations laid down at *page 156*, in order to obtain from thence a farther correction of the value

($e \times \left\{ \begin{array}{l} 1 - B \cos. \beta z + 0,007186 \cos. pz - 0,1869 B \cos. \overline{p-\beta.z} \\ + 0,004 B \cos. \overline{p+\beta.z} - 0,000023 \cos. 2pz + 0,0008 B \cos. 2\overline{p-\beta.z} \end{array} \right\}$) above found, the several powers of the series $B \cos. \beta z - 0,007186 \cos. pz + 0,1869 B \cos. \overline{p-\beta.z} \&c.$ (before omitted) must be now taken, or such terms thereof, at least, as are of consequence enough to merit regard. Thus, in the square, or second power, the terms which appear considerable enough to merit an examination, at least, will be those arising from the squares and the double rectangles of the three first terms of the root, which are vastly larger than the others. These will be $\frac{1}{2} B^2 + \frac{1}{2} B^2 \cos. 2\beta z - ,007186 B \cos. \overline{p-\beta.z} - ,007186 \cos. \overline{p+\beta.z} + 0,1869 B^2 \cos. \overline{p-2\beta.z} + 0,1869 B^2 \cos. pz + ,000025 + ,000025 \cosine 2pz - ,00135 B \cosine \beta z - ,00135 B \cos. 2\overline{p-\beta.z} + ,0175 B^2 + ,0175 B^2 \cos. 2\overline{p-2\beta.z}$. But, in the value of $\frac{6}{1-w}^{-3}$, these terms are affected with the common multiplicator $\frac{6}{p^2}$ (*vide p. 154*); and, in the value of

$\overline{1-w}^{-4}$, by the common multiplicator $\frac{10}{e^4}$: so that, in order to find the effect of the first of them ($\frac{1}{2}B^1$) from the formula at p. 156, we must compare $\frac{1}{2}B^1 \times \frac{6}{e^2}$ and $\frac{1}{2}B^1 \times \frac{10}{e^4}$, with $\frac{M}{e^3} \times \cos. \alpha z$, and $\frac{N}{e^4} \times \cos. \alpha z$, respectively; whence $M = 3B^1$, $N = 5B^1$, and $\alpha = 0$: and consequently,

$$\frac{P}{e^4 f} \times \frac{N}{p-a} + \frac{MP'}{2P} - \frac{\alpha M}{6} \times \frac{\cos. \overline{p-a.z}}{1-\overline{p-a}^2} +$$

$$\frac{P}{e^4 f} \times \frac{N}{p+a} + \frac{MP'}{2P} + \frac{\alpha M}{6} \times \frac{\cos. \overline{p+a.z}}{1-\overline{p+a}^2} \&c. = ,029 B^1 \cos. pz.$$

In the same manner, with respect to the second term, $\frac{1}{2}B^2 \cos. 2\beta z$, we have $M = 3B^2$, $N = 5B^2$ (as before), and $\alpha = 2\beta$; from whence the correction, answering thereto (exclusive of the general factor e) will come out, $,03178 B^2 \cos. \overline{p-2\beta.z} + ,0023 B^2 \cos. \overline{p+2\beta.z} + ,0028 \cos. 2\beta z$.

Again, in relation to $-,007186 B \cos. \overline{p-\beta.z}$, we have $M = -,431B$, $N = -,0718B$, and $\alpha = p-\beta$; from which will be found $,000735 B \times \frac{\cos. \beta z}{1-\beta\beta} -,000073 B \cos. \overline{2p-\beta.z} + ,000464 B \cos. \overline{p-\beta.z}$; where the first term is of that species on which the motion of the *apogee* depends, and where the *secund* is too small to be farther regarded.

In like sort, from $-,007186 B \cos. \overline{p+\beta.z}$, we shall get $-,000060 B \times \frac{\cos. \beta z}{1-\beta\beta}$, besides two other terms; which, on account of their extreme smallness, may be intirely neglected.

By proceeding on, in this manner, two, or three small terms, more, (producing equations of a few seconds, each) will be found; which being joined to those above, we shall from thence have, $+ ,03 B^1 \cos. pz + ,00055 B \cos. \overline{p-\beta.z} + ,03028 B^2 \cos. \overline{p-2\beta.z} + ,0072 B^2 \cos. 2\beta z + ,0070 B^2 \cos. \overline{2p-2\beta.z} + ,00016 B \times \frac{\cos. \beta z}{1-\beta\beta}$; which, drawn into the general multiplicator e , will give the whole increase of $1-w$, by taking in the square of the series aforesaid. As to the increase arising from the

the

the cube thereof, few of the terms will be found considerable enough to deserve notice, the only ones of any consequence to be collected from thence, being $-\frac{00004B}{1} + \frac{0,7B^1 \times \cos p-2\beta.z}{1-\beta\beta}$
 $-0,11B^1 \times \cos p-3\beta.z - \frac{08B^1 \times \cos 2p-3\beta.z}{1-\beta\beta}$
 $-\frac{055B^1 \times \frac{\cos \beta.z}{1-\beta\beta}}$: But, from the *biquadrate*, we shall only have one term ($1,1B^4 \cos p-2\beta.z$) that can produce an equation of more than a *second*, or two; though the effect of this one term will, alone, amount to near a *minute*; which is owing to the smallness of the exponent $p-2\beta$, consonant to what has been before inculcated (at page 157). And it may be worth while to take farther notice here, that all the terms hitherto determined, except two or three, belong to one, or the other of the two *kinds* there specified.

If, now, the several quantities above brought out, be collected into one sum, we shall have $\frac{0,03B^1 \cos pz + \frac{00051B}{1} - 0,7B^1 \times \cos p-\beta.z + \frac{0,3028B^2 + 1,1B^4 \times \cos p-2\beta.z}{1} + \frac{0,072B^2 \cos 2\beta.z + 0,070B^1 \cos 2p-2\beta.z - 0,11B^1 \cos p-3\beta.z}{1} - \frac{08B^1 \cos 2p-3\beta.z + \frac{00016B}{1} - \frac{055B^1 \times \frac{\cos \beta.z}{1-\beta\beta}}{1}}$; of all which terms, $\frac{0,3028B^2 + 1,1B^4 \times \cos p-2\beta.z}{1}$ is by far the greatest, and so considerable as to produce other terms, or corrections, of consequence enough to be regarded here. The new terms arising from the entering of this term into the first power of the series, for the value of $1-w$, will appear to be $-\frac{0032B^1 \cos 2\beta.z - 0047B^1 \cos 2p-2\beta.z + 0026B^2 \cos p-2\beta.z}{1}$ (which are found as above, by making $M=3 \times -0,3028B^2$, $N=4 \times -0,3028B^2$, $\alpha=p-2\beta$, and neglecting $1,1B^4$): And by the entering thereof into the square of that series, from the double rectangle under it and the first term, there will be had $+\frac{088B^1 \cos 2p-3\beta.z}{1} - \frac{019B^1 \cos p-3\beta.z + \frac{03B^1 \times \frac{\cos \beta.z}{1-\beta\beta}}{1}}$: which, together with the quantities before found, being joined to the former value of $1-w$, we have at length

$$1-\pi=\epsilon \times \left\{ \begin{array}{l} 1-B \cos . \beta z + \frac{.007186 + .03 B^2 \times \cos . p z - .0,1864 B + .0,7 B^2 \times \cos . p - \beta . z}{.00023 \cos . 2 p z} \\ + .004 B \cos . p + \beta . z + .0,3054 B^2 + 1,1 B^2 \times \cos . p - 2 \beta . z - .000023 \cos . 2 p z \\ + .0008 B \cos . 2 p - \beta . z + .004 B^2 \cos . 2 \beta z + .0023 B^2 \cos . 2 p - 2 \beta . z \end{array} \right. \left[+ .0,13 B^2 \cos . p - 3 \beta . z \right]$$

In which equation the terms of the species, $\cos . \beta z$, are not put down; because the original term $-B \cos . \beta z$, which is equivalent to them, is still retained.

Those terms, however, though of no use in this equation, are not to be intirely disregarded; since on them the motion of the *apogee* depends. By collecting them, and the former value of $-B$, into one sum, we get the equation $-B = -\frac{.01619 B}{1-\beta \beta}$

$$+ \frac{.00016 B - .055 B^2}{1-\beta \beta} + \frac{.03 B^2}{1-\beta \beta} = -\frac{.01603 B + .025 B^2}{1-\beta \beta} : \text{from whence,}$$

when B is assigned, the value of β will be found, but *still* something short of the true value, as given from *observation*.

But it ought to be now remembered, that all the equations hitherto brought out, are derived on the hypothesis, that the *true motion* of the sun is to *that* of the moon, in a constant proportion: which is near the truth, only; since the distance of the sun from the moon, whereon the perturbing forces depend, is sometimes greater, or less by two degrees, than according to that hypothesis: which difference must, of consequence, render other corrections necessary. How *these* may be introduced I shall here instance, by first considering the motion of the earth about the sun, as uniformly performed in a circular orbit; leaving, to be determined by a future operation, the other equations arising from the eccentricity and parallax, which will be obtained in the same manner.

It is found at page 159, that the moon's *mean motion*, answering to the *true motion* z , is rightly represented by $z + \frac{(B) \times \sin . \beta z}{b \beta} + \frac{(C) \times \sin . \gamma z}{b \gamma} + \frac{(D) \times \sin . \delta z}{b \delta} \&c$; from whence the *mean*, or *true motion* of the sun (for, in a circular orbit, they are the

$$\text{same) will be had} = m \times z + \frac{(B) \times \sin . \beta z}{b \beta} + \frac{(C) \times \sin . \gamma z}{b \gamma} \&c.$$

$$= m z + m z', \text{ supposing } z' = \frac{(B) \times \sin . \beta z}{b \beta} + \frac{(C) \times \sin . \gamma z}{b \gamma} \&c. \text{ Hence}$$

the

the true distance of the moon from the sun, will be $z - mz - mz'$, and the double thereof equal to $1 - m \times 2z - 2mz'$, or to $pz - 2mz'$: of which the sine is $\sin.pz \times \cos.2mz' - \cos.pz \times \sin.2mz'$; but the arch $2mz'$ being exceeding small, the sine thereof will be nearly equal to the arch itself, and the cosine very nearly equal to the radius, or unity; and consequently the sine of the double distance of the moon from the sun $\sin.pz - \cos.pz \times 2mz' = \sin.pz - \cos.pz \times \frac{2m(B) \times \sin.\beta z}{b\beta}$

$$\&c. = \sin.pz + \frac{m(B)}{b\beta} \times \frac{\sin.p - \beta.z - \sin.p + \beta.z}{b\gamma} + \frac{m(C)}{b\gamma} \times \frac{\sin.p - \gamma.z - \sin.p + \gamma.z}{b\gamma} \&c.$$

In like manner, the co-sine thereof will be had ($= \cos.pz \times \cos.2mz' + \sin.pz \times \sin.2mz' = \cos.pz + \sin.pz \times 2mz'$ nearly) $= \cos.pz + \frac{m(B)}{b\beta} \times \cos.p - \beta.z - \cos.p + \beta.z + \frac{m(C)}{b\gamma} \times \cos.p - \gamma.z - \cos.p + \gamma.z \&c.$

Which two quantities being multiplied by P, it will appear, that the new terms arising in the two expressions of the forces (besides those, P sin.pz and P cos.pz. in the former hypothesis) are $\frac{mP(B)}{b\beta} \times \sin.p - \beta.z - \sin.p + \beta.z + \frac{mP(C)}{b\gamma} \times \sin.p - \gamma.z - \sin.p + \gamma.z \&c.$ and $\frac{mP(B)}{b\beta} \times \cos.p - \beta.z - \cos.p + \beta.z + \frac{mP(C)}{b\gamma} \times \cos.p - \gamma.z - \cos.p + \gamma.z \&c.$

Now it will appear, by a comparison with the formula on p. 157, that the effect of the two first, corresponding terms here exhibited, making $\pi = p - \beta$, will be $\frac{mP(B)}{b\beta e^4 f} \times \frac{2}{\pi} + 1 \times \frac{\cos.\pi z}{1 - \pi\pi} + \frac{4}{\pi - \beta} + \frac{1}{2} - \frac{\beta}{2} - \frac{1 - \beta}{\pi} \times \frac{mPB(B)}{b\beta e^4 f} \times \frac{\cos.\pi - \beta.z}{1 - \pi - \beta} + \frac{4}{\pi + \beta} + \frac{1}{2} + \frac{\beta}{2} - \frac{1 - \beta}{\pi} \times \frac{mPB(B)}{b\beta e^4 f} \times \frac{\cos.\pi + \beta.z}{1 - \pi + \beta} \&c.$ which, because $m = .0748$, $\frac{P}{e^4 f} = -.0084$, and $b = 1$, will be equal to

$$-\frac{,00063(B)}{\beta} \text{ into } \frac{2}{\pi} + 1 \times \frac{\text{cof. } \pi z}{1-\pi\pi} + \frac{4}{\pi-\beta} + \frac{1}{2} - \frac{\beta}{2} \frac{1-\pi\beta}{\pi} \\ \times \frac{\text{Bcof. } \frac{\pi-\gamma.z}{1-\pi-\beta}}{\frac{1}{\pi+\beta}} + \frac{4}{\pi+\beta} + \frac{1}{2} + \frac{\beta}{2} - \frac{1-\gamma\gamma}{\pi} \times \frac{\text{Bcof. } \frac{\pi+\gamma.z}{1-\pi+\beta}}{\frac{1}{\pi+\beta}} + \\ \frac{4}{\pi-\gamma} + \frac{1}{2} - \frac{\gamma}{2} - \frac{1-\gamma\gamma}{\pi} \times \frac{\text{Ccof. } \frac{\pi-\gamma.z}{1-\pi-\gamma}}{\frac{1}{\pi-\gamma}} + \&c. = -,0081 (B)$$

$\times \text{cofine } \overline{p-\beta.z} + ,0187B (B) \times \text{cof. } \overline{p-2\beta.z} + ,0011B (B) \times \text{cof. } pz \&c.$ where all the terms, after the two first, may, on account of their extreme smallness, be intirely neglected.

In the very same manner, expounding π by $p+\beta$, $p-\gamma$ ($=0$), $p+\gamma$ ($=2p$), $p-\delta$ ($=\beta$), and $p+\delta$ ($=2p-\beta$), successively, and substituting $+\frac{,00063(B)}{\beta}$, $-\frac{,00063(C)}{\gamma}$, $+\frac{,00063(C)}{\gamma}$,

$-\frac{,00063(D)}{\delta}$, and $+\frac{,00063(D)}{\delta}$, respectively, in the room of

$-\frac{,00063(B)}{\beta}$; there will, in the first case, be found the term

$-,00015 (B) \times \text{cof. } \overline{p+\beta.z}$; being the only one producing an equation of more than a single second; in the second and third cases, no term at all, worth notice, will be found: but, in the fourth (where $\pi=\beta$) two pretty considerable ones, $,00222 (D) \times \frac{\text{cof. } \beta z}{1-\beta\beta}$, and $-,0233 D(D) \times \text{cof. } \overline{p-2\beta.z}$, do arise; whereof

the former is of that species on which the motion of the apogee depends, and gives, an increase of that motion, of about $\frac{1}{20}$ part. By pursuing the same method, the effect of the second order of terms, $\frac{(BB) \times \text{fin. } 2\beta z}{b, 2\beta}$, $\frac{(BC) \times \text{fin. } \overline{\gamma-\beta.z}}{b, \gamma-\beta}$ &c. may also be comput-

ed; but here it will be sufficient to make use of the first term of the general formula alone: from whence are obtained the quantities $+,0045 (BB) \times \text{cof. } \overline{p-2\beta.z}$, and $-,0059 (DD) \times \text{cof. } \overline{p-2\beta.z}$; which are both of that kind, specified at p . 157, rendered considerable by the smallness of their exponents, and are the only ones here, that merit regard. These being, therefore, joined to those found above, the whole correction sought will be, $-,00081(B) \times \text{cof. } \overline{p-\beta.z} - ,00015(B) \times \text{cof. } \overline{p+\beta.z}$

$+ 0,0187B(B) - 0,233D(D) + 0,0045(BB) - 0,0059(DD)$
 $\times \text{cof. } p - 2\beta.z$. But (B) being $= 2B$, $(D) = 2D$ $(BB) = \frac{1}{2}B^2$,
 and $(DD) = \frac{1}{2}D^2$, nearly, *vid. p. 159* ($\bar{B}$, $\bar{C}$, $\bar{D}$, &c. being
 neglected, as too inconsiderable to be regarded here) our *expressi-*
on may therefore be changed to $- 0,0162 B \text{ cof. } p - \beta.z$
 $- 0,0003 B \text{ cof. } p + \beta.z + 0,0441 B^2 - 0,0554 D^2 \times \text{cof. } p - 2\beta.z$
 which being added to the value of $1 - w$, before found, we
 thence have

$$1 - w = ex \left\{ \begin{array}{l} 1 - B \text{ cof. } \beta z + 0,007186 + 0,03B^2 \times \text{cof. } p z - 0,2026B + 0,7B^2 \times \text{cof. } p - \beta.z \\ + 0,0037 B \text{ cof. } p + \beta.z + 0,03495B^2 - 0,0554D^2 + 1,1B^2 \times \text{cof. } p - 2\beta.z \\ - 0,00023 \text{ cof. } 2pz + 0,0008 B \text{ cof. } 2p - \beta.z + 0,004B^2 \text{ cof. } 2\beta z + \\ 0,0023B^2 \text{ cof. } 2p - 2\beta.z + 0,13B^2 \text{ cof. } p - 3\beta.z \end{array} \right.$$

And, by writing the coefficient of the term $- \frac{0,00222(D) \times \text{cof. } \beta z}{1 - \beta\beta}$,
 in the equation for the motion of the *apogee*, it will here be-
 come $- B = - \frac{0,01603B + 0,025B^2}{1 - \beta\beta} - \frac{0,00222(D)}{1 - \beta\beta} = -$

$\frac{0,01693B + 0,028B^2}{1 - \beta\beta}$; because $(D) = 2D = 2 \times 0,2026B + 0,7B^2$;

whence $1 - \beta\beta = 0,01693 + 0,00028B^2 = 0,017015$ (supposing
 $B = 0,05505$); and consequently $1 - \beta = 0,00854$; which
 value is, now, about $\frac{1}{100}$ part greater than the true value, given
from observation, and is near enough to shew, that the
 force of the sun is sufficient to produce all the motion of the
 moon's *apogee*, without supposing a change in the general law
 of gravitation, from the inverse ratio of the squares of the dis-
 tances. Several very small terms belonging to this equation,
 have been omitted, besides those arising from the sun's *excentri-*
city, and the *inclination* of the *lunar orbit*; which together,
 may very well be supposed, sufficient to cause a difference equal
 to that abovementioned.

If, now, the value of, B expressing the *mean excentricity*,
 in *Sir Isaac Newton's* lunar theory, be expounded by $0,05505$,
 according to that author, the general equation of the orbit,
 found above, will here become

The Resolution of some General Problems

$$1-u=\epsilon x \begin{cases} 1-0,05505 \text{ cof. } \beta z + 0,007276 \text{ cof. } pz - 0,01127 \text{ cof. } \overline{p-\beta.z} \\ + 0,000204 \text{ cof. } \overline{p+\beta.z} + 0,0010623 \text{ cof. } \overline{p-2\beta.z} - 0,000023 \text{ cof. } 2pz \\ + 0,000044 \text{ cof. } 2p-\beta.z + 0,000012 \text{ cof. } 2\beta z + 0,000007 \text{ cof. } 2p-2\beta.z \\ + 0,000021 \text{ cof. } p-3\beta.z \end{cases}$$

But this value must now be corrected by the difference arising from our having, in all the preceding calculations, taken the divisor $bb=1$, instead of the true value $1+\frac{1}{2} \times BB+CC+DD&c.$ (*vid. p. 159.*) which value, because $B=0,05505$, $C=-0,007276$ &c. is given $=1,0097$. From whence and the equations, on *p. 153* and *163*, it appears, that all the terms above exhibited, in whose exponents the quantity p is, *singly*, concerned, ought to be diminished in the ratio of 1 to 1,0097; and that all *those*, where $2p$ is in like manner concerned, ought to be diminished, in the duplicate of that ratio: by which means our equation is, at length, reduced to

$$1-u=\epsilon x \begin{cases} 1-0,05505 \text{ cof. } \beta z + 0,007206 \text{ cof. } pz - 0,01116 \text{ cof. } \overline{p-\beta.z} \\ + 0,000202 \text{ cof. } \overline{p+\beta.z} + 0,001052 \text{ cof. } \overline{p-2\beta.z} - 0,000022 \text{ cof. } 2pz \\ + 0,000043 \text{ cof. } 2p-\beta.z + 0,000012 \text{ cof. } 2\beta z + 0,000007 \text{ cof. } 2p-2\beta.z \\ + 0,000021 \text{ cof. } p-3\beta.z \end{cases}$$

From whence all the great *equations* of the moon's motion, and all the smaller ones, except those depending on the sun's *eccentricity* &c. are obtained, within less than half a *minute* of the truth; supposing the mean *eccentricity* (B) to be here truly assigned. If it should be found necessary to augment, or diminish the value thereof, then the term $-0,01116 \text{ cof. } \overline{p-\beta.z}$ (producing the *equation*, called the *evection*) must be also augmented or diminished, in the same ratio; and the term $+0,001052 \text{ cof. } \overline{p-2\beta.z}$ (which is the next considerable of those wherein β enters) must be augmented or diminished in the duplicate of that ratio. As to the rest of the terms, they are so small, that a little alteration in the value of B will produce no difference in them worth notice.

In the same manner, the inequalities caused in the moon's motion by the sun's *eccentricity*, may be computed. For the mean motion of the moon being given, very nearly, by the preceding calculations, the mean motion of the sun, being in proportion thereto as m to 1, will be also known; from whence, and the *eccentricity*, the sun's *true anomaly*, and distance from the

the earth will be had, in a series of sines and co-sines of the multiples of the arch z ; whose exponents and coefficients are all intirely known: by means of which, the sun's *true distance* from the moon being (nearly) obtained, the *new* terms arising from the eccentricity, in the general expressions for the perturbing forces, will be had, in consequence thereof; and, lastly, the effects themselves, produced thereby in the general equation of the orbit.

When the equation of the orbit, or the value of $1-w$ (the reciprocal of the moon's distance from the earth) is thus determined to a sufficient degree of exactness (by repeating the operation if necessary), the difference between the *true*, and *mean motions* of the moon, must, from thence, be found, in terms of the latter: this may be done by first finding the *mean motion* in terms of the *true* (as is shewn at p. 159) and then reverting the series; or, otherwise, without finding the *mean motion* at all, by the resolution of the fluxionary equation

$$\dot{t} = \frac{g}{2c} \times \frac{z}{\Sigma 1 \times 1 - u^2}$$
; according to the method made use of of in a former *paper*, in this collection. The process (which is more laborious than difficult) I shall not insert here, as my design, in this *miscellaneous work*, is not to exhibit every operation necessary to the forming of a *complete* theory of the moon's motion; which, for its importance, and the great variety and intricacy of calculations arising therein, may very well merit to be the subject of a *volume*, by itself.

It may suffice, in this place, to have pointed out (by a method not very perplexed) how the different *inequalities* of that motion may be determined, and the moon's true place, *according to gravity*, ascertained. At another time, if health permits, I may, not only give, at large, the application of the *equations* and *precepts* here delivered, but also a new set of *lunar tables*, deduced therefrom; which (though a work of much labour) I shall the more chearfully undertake, as *Dr. Bradley* has very obligingly offered to assist me with any observations, that may be wanting in order to the compleating of the design, and establishing the *theory* on a proper *basis*. And I have some reason to hope, that, when that part of the *data*, which can be only known from observations, is truly settled, the

the place of the moon may be *always* obtained, within about a *minute* of the truth, even, without the use of a multitude of tables, by means of proper contrivances, respecting the smaller equations, or the less considerable terms of the general series.

I shall conclude what I have to say on this subject, at present, with obviating a difficulty in the application of the equations above-mentioned, when the effect in the moon's motion, depending on the sun's *eccentricity* and *parallax* *, are computed thereby. In this case, a new species of (very small) terms, affected with the co-sine of the arch z , will be found to enter into the general equation (*on page 151*) whereof the effect cannot be determined in the same manner with that of the other terms, affected with the co-sines of the multiples of that arch: For if, according to the method of proceeding there laid down, we assume a term (as $1 - \pi\pi \cdot \Pi \cos. \pi z$) in the upper line, $1 - \beta\beta \cdot B \cos. \beta z + 1 - \gamma\gamma \cdot C \cos. \gamma z + 1 - \delta\delta \cdot D \cos. \delta z$ &c. in order to compare it with a term, $g \cos. z$, of the aforesaid species, we

* To determine the perturbing forces, so as to take in the effect of the sun's *eccentricity* and *parallax*; let $a = CS$, $x = CB$, and $y = CF$; then BS being $= \sqrt{aa - 2ay + xx}$, the force $k \times \frac{CS^2 \times CB}{BS^3}$ in the direction CB (*vid. p.*

160) will be $= \frac{kx}{a} \times 1 - \frac{2y}{a} + \frac{xx}{aa} \Big)^{-\frac{1}{2}} = \frac{kx}{a} \times 1 + \frac{3y}{a}$ &c. and that $(k \times \frac{CS^3}{SB^3}$

$-k)$ in the direction BH , equal to $k \times 1 - \frac{2y}{a} + \frac{xx}{aa} \Big)^{-\frac{1}{2}} - k = k \times$

$\frac{3y}{a} - \frac{3xx}{2aa} + \frac{15yy}{2aa}$ &c. which, because $y = x \times \cos. BCS$, will be reduced to

$\frac{kx}{a} \times 1 + \frac{3x}{a} \times \cos. BCS$ &c. and $k \times \frac{3x}{a} \times \cos. BCS + \frac{xx}{4aa} \times 9 + 15 \cos. 2BCS$ &c. respectively. But the last of these forces may, again, be resolved into two others; the one in the direction BC , expressed by $k \cos. BCS \times$

$\frac{3x}{a} \times \cos. BCS + \frac{xx}{4aa} \times 9 + 15 \cos. 2BCS$, ($= \frac{3kx}{2a} \times 1 + \cos. 2BCS +$

$\frac{3kxx}{8aa} \times 1 + \cos. BCS + 5 \cos. 3BCS$); and the other, in a direction perpendicular

thereto, expressed by $k \sin. BCS \times \frac{3x}{a} \times \cos. BCS + \frac{xx}{4aa} \times 9 + 15 \cos. 2BCS$

(=

we shall then have $\pi=1$, and consequently $1-\pi\pi=0$; and so, the whole $(1-\pi\pi.\Pi \cos.\pi z)$ intirely vanishing, we have nothing left to compare with the term proposed ($g \cos. z$) whereby its effect can be known. Nevertheless, it is by means of a term ($\Pi \cos. \pi z$) of the same species, entering into the series for the value of $1-w$, that the effect in question must be exhibited: for, though indeed the term $\Pi \cos. \pi z$ (or $\Pi \cos. z$), by receiving the co-efficient $1-\pi\pi (=0)$ intirely vanishes, in the first line of the general equation, yet, that will not be the case in the other parts of the equation; wherein it will be found, affected in the same manner with the term $B \cos. \beta z$ (whereof the determination hath been given, in computing the motion of the *apogee*); which must necessarily appear to be the case, if it be considered, that the terms $\Pi \cos. \pi z$ and $B \cos. \beta z$ are alike concerned in the original series $1-\Pi \cos. \pi z-B \cos. \beta z-C \cos. \gamma z$ &c. from whence the equation itself is derived. Hence we not only have a proper term to compare with the given one ($g \cos. z$), but also an easy way to discover what the result will

$$(\frac{3kx}{2a} \times \sin. 2BCS + \frac{3xx}{8aa} \times \sin. BCS + 5 \sin. 3BCS). \text{ From the former}$$

of which, let the force $\frac{kx}{a} \times 1 + \frac{3x}{a} \times \cos. BCS$. in the opposite direction be

$$\text{subtracted, and the remainder, } \frac{kx}{2a} \times 1 + 3 \cos. 2BCS + \frac{3kxx}{8aa} \times$$

$3 \cos. BCS + 5 \cos. 3BCS$, will be the force whereby the gravity of the moon to the earth, is diminished. But the quantity k (which must be exterminated) is, at the mean distance (d) of the sun and earth, found to be equal to $d \times$

$$\frac{m^2 e^4 f}{bb \times CA} \text{ (vid. p. 162); whence, } a^2 : d^2 :: d \times \frac{m^2 e^4 f}{bb \times CA} : \frac{d^3}{aa} \times \frac{m^2 e^4 f}{bb \times CA} = \text{ the}$$

general value of k ; which being substituted instead thereof, the force in the direction of the *radius vector* will become

$$\frac{m^2 e^4 f}{2bb} \times \frac{d^3}{a^3} \times \frac{CB}{CA} \times 1 + 3 \cos. 2BCS + \frac{CB}{CS} \times \frac{3}{4} \cos. BCS + \frac{15}{4} \cos. 3BCS; \text{ and}$$

that in the *perpendicular direction*,

$$= \frac{m^2 e^4 f}{2bb} \times \frac{d^3}{a^3} \times \frac{CB}{CA} \times 3 \sin. 2BCS + \frac{CB}{CS} \times \frac{3}{4} \sin. BCS + \frac{15}{4} \sin. 3BCS:$$

Where $m^2=0.005595$, and $bb=1.0097$, nearly (vid. p. 174); and where no regard need be had to the value of $e^4 f$, as this quantity, when substitution is made in the general formula (en p. 157.) intirely vanishes.

A a

bc;

be; for seeing that $\Pi \text{ cof. } \pi z$ and $B \text{ cof. } \beta z$ have *there* the same coefficients, and that the sum of those belonging to $B \text{ cof. } \beta z$ has been already found, in computing the motion of the *apogee*, to be $= -\frac{1}{1-\beta\beta}$, it follows, that $-\frac{1}{1-\beta\beta} \Pi \text{ cof. } \pi z$ (or $-\frac{1}{1-\beta\beta} \Pi \text{ cof. } z$) will express the value of all the terms of this species, arising in the equation from the introduction of $\Pi \text{ cof. } z$ into the original, or assumed series: whence, by comparing this quantity with the given one, $g \text{ cof. } z$ (that is, by making $-\frac{1}{1-\beta\beta} \Pi \text{ cof. } z + g \text{ cof. } z = 0$) we get $\Pi = \frac{g}{1-\beta\beta}$; and consequently $\Pi \text{ cof. } z = \frac{g}{1-\beta\beta} \times \text{cof. } z$; whereby the required effect of the quantity $g \text{ cof. } z$, is known.

'Tis true, indeed, that the new term $\Pi \text{ cof. } z$, thus determined, will introduce an infinity of others; but, of these, none will be considerable enough to merit the least degree of attention, except (perhaps) such as are expressed by the co-sine of $1-\beta.z$; which, because of the smallness of $1-\beta$, may, for reasons before specified, produce an effect not to be neglected without a proper examination. To give here the quantity of this effect, it will be necessary to observe, that, of all the terms in the values of $\frac{1}{1-w}^{-3}$ and $\frac{1}{1-w}^{-4}$, *that* which arises by the multiplication of $\frac{10\Pi}{e^4} \times \text{cof. } z$ into $2D \times \text{Cof. } \delta z$, and comes from

the part $\frac{10}{e^4} \times \Pi \text{ cof. } z + B \text{ cof. } \beta z + C \text{ cof. } \gamma z + D \text{ cof. } \delta z \text{ \&c.}$ will so much exceed, in its effect, the others wherein Π is concerned, as to render them, in a manner, inconsiderable. But this term $\frac{10\Pi}{e^4} \text{ cof. } z \times 2D \text{ cof. } \delta z$ (or $\frac{10PD}{e^4} \text{ cof. } z \times 2 \text{ cof. } \overline{p-\beta.z}$) is

resolved into $\frac{10PD}{e^4} \times \text{cof. } \overline{p-\beta+1.z}$, and $\frac{10PD}{e^4} \times \text{cof. } \overline{p-\beta-1.z}$; whereof the former part only, in which the difference of β and 1 is concerned, is for our purpose: from whence (as appears by the observations at *p. 156*) a new term expressed by $\frac{eP}{e^4 f} \times \frac{10PD}{\beta-1} \times \frac{\text{cof. } \overline{\beta-1.z}}{1-\beta-1}$ (or $\frac{eP}{e^4 f} \times \frac{10PD}{1-\beta} \times \text{cof. } \overline{\beta-1.z}$, nearly) will arise in the required value of $1-w$. Of the two terms here found, the former gives a very small equation depending

ing on the moon's distance from the sun's *apogee*; and the latter, an equation something more considerable, whose *argument* is the distance of the moon's *apogee* from *that* of the sun.

By having shewn above, that the effect of such terms, or forces, as are proportional to the *co-sine of the arch* z , is explicable by means of the co-sines of that arch, and of its multiples, (no less than the effects of the other terms that are proportional to the *co-sines of the multiples thereof*) a very important point is determined: For, since it appears thereby, that no terms enter into the equation of the orbit but what by a regular increase and decrease, do after a certain time return again to their former values, it is evident from thence, that the mean motion, and the greatest quantities of the several equations, undergo no change from *gravity*.

THE END.

ERRATA.

Page 5 line 18, for R read R'; l. 21, for R' r. R; p. 30. l. 9, for $ba+bb$ r. $ba+b$; l. 13, for t r. b; p. 39. l. 14, put the *comma* before $\gamma O \Delta$; p. 58. l. 11, for PSO r. PSC; p. 59. l. 15, for $\frac{c^3 z}{2a} r. \frac{cz}{a}$; l. 22, for Fr. N; p. 114, l. 32. r. powers of; p. 127, l. 9, for AGO r. AOG; p. 128. l. 30, for BL. BR; p. 131. l. 23, for BC r. nC; p. 138. l. 15, for Eef r. Ee; p. 143, l. 15 dele *to*; p. 152, l. 2, for $\beta\beta$, r. $\gamma\gamma$; p. 160. l. 27, for $\frac{BC}{SC} r. \frac{BC}{SD}$.
In fig. 18, at the intersection of the circumference of the circle and the right-line Fm, place an ϵ .

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Fig. 1.

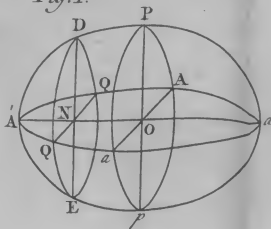


Fig. 2.

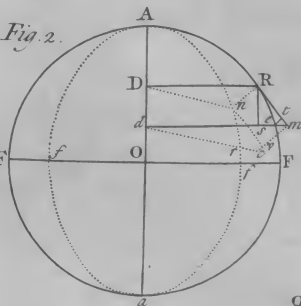


Fig. 3.

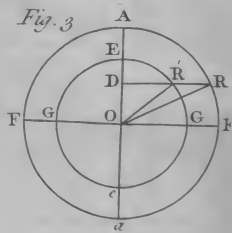


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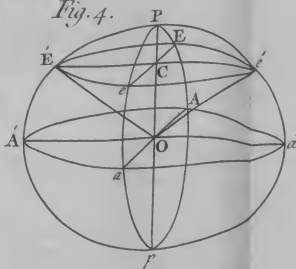


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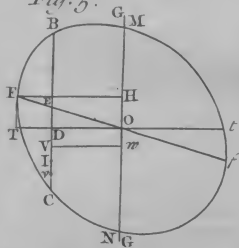


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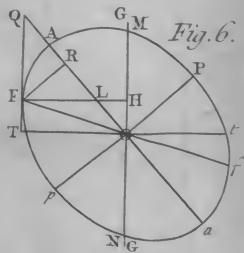


Fig. 7.

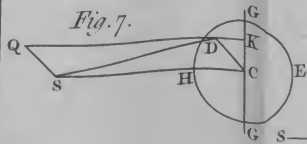


Fig. 8.

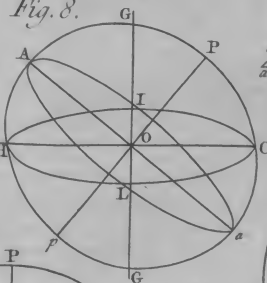


Fig. 9.

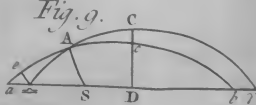


Fig. 10.

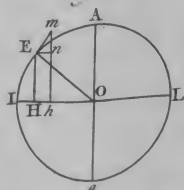


Fig. 11.

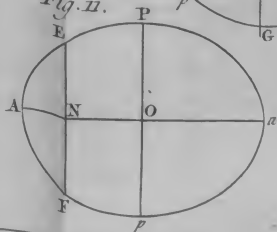
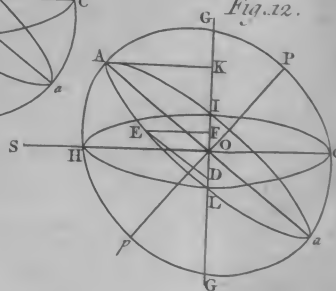


Fig. 12.



1800

1801

1802

1803

